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SCATTERING OF ELASTIC WAVES IN A PERTURBED ISOTROPIC HALF SPACE WITH A FREE BOUNDARY. THE LIMITING ABSORPTION PRINCIPLE.

by

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Abstract

In this article we consider the selfadjoint operator governing the propagation of elastic waves in a perturbed isotropic half space with a free boundary condition. We prove the limiting absorption principle in appropriate Hilbert spaces for this operator. We also prove decreasing properties for the eigenfunctions associated with strictly positive eigenvalues of this operator.

The proofs are based on the limiting absorption principle for the selfadjoint operator governing the propagation of elastic waves in an homogeneous isotropic half space with a free boundary and on the so called division theorem for it. Both perturbations of $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R};x_2>0\}$ and $\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \in \mathbb{R}^3;x_3>0\}$ are studied.

Résumé

Dans cet article on considère l'opérateur autoadjoint associé à la propagation des ondes élastiques dans un demi espace isotrope perturbé avec la condition de surface libre sur le bord. On démontre le principe d'absorption limite pour cet opérateur. On démontre aussi des propriétés de décroissance des fonctions propres associées aux valeurs propres strictement positives de cet opérateur.

Les démonstrations précédentes reposent sur le principe d'absorption limite pour l'opérateur autoadjoint associé à la propagation des ondes élastiques dans un demi-espace isotrope homogène avec la condition de surface libre sur le bord et sur un théorème de division pour cet opérateur. On considère les deux demi-espaces $\mathbb{R}^2_+ = \{(x_1,x_2);x_2>0\}$ et $\mathbb{R}^3_+ = \{(x_1,x_2,x_3);x_3>0\}$.

Key words

Elastic waves - Perturbed isotropic half space with a free boundary - The limiting absorption principle.

Mots clés

Ondes élastiques - Demi-espace isotrope perturbé avec la condition de surface libre - Le principe d'absorption limite.

I. INTRODUCTION

This article is the first one of two papers concerned with scattering of elastic waves in a perturbed isotropic half space with a free boundary. More precisely we consider perturbations with compact support of an homogeneous isotropic elastic half space with a free boundary. The perturbations can be bounded obstacles, solid or fluid bounded inclusions, cracks of finite dimensions or an inhomogeneous part of the material.

This problem has its own interest. Usually in non-destructive evaluation of solids one of the main problems is the scattering of elastic waves by a crack of finite dimensions located at the plane (or curved) interface of two isotropic media. In fact the real problem is the inverse scattering one, i.e., the determination of the size, shape and orientation of the crack. Nevertheless the direct scattering problem is a necessary prerequisite for solving the inverse scattering one.

Scattering of elastic waves by cracks has been studied by asymptotic methods in [2]. In this article this problem is studied from the point of view of the mathematical theory of scattering (see [24], [27], [30]). From this point of view the difference between the case of a perturbed isotropic half space with a free boundary and that of two isotropic half spaces with a perturbed plane interface is not very important. The case of a perturbed isotropic half space with a free boundary is simpler. That is the reason why we begin with it here.

In this article we essentially prove the so called limiting absorption principle for the selfadjoint operator governing the propagation of elastic waves in a perturbed isotropic half space with a free boundary. This selfadjoint operator is denoted by A. In the case of a homogeneous isotropic half space with a free boundary the selfadjoint operator is denoted by A_0.

The limiting absorption principle is a stationnary method used in the spectral theory of selfadjoint operators associated with the partial differential equations of the mathematical physics.

To prove the limiting absorption principle we used the methods of S. Agmon [3] (see also [18], [23], [25], [27]), of D.M. Eidus ([12], [13]; see also [30]) and of R. Phillips [22] (see also [20]). The key result is a division theorem for A_0 (see theorem 4.1). An another application of the division theorem we give in this article is a result concerning decreasing properties of eigenfunctions of A associated with positive eigenvalues (see theorem 5.5).

In fact we adapt to linear elasticity the method that has been used in [6] for sound propagation in stratified media (see also [29]).

The starting point is the explicit knowledge of the modes for an homogeneous isotropic halfspace with a free boundary. We call them the unperturbed modes. These modes are the P-plane waves and their reflections, the SV and SH plane waves and their reflections and the Rayleigh surface waves (see [1], [4], [5], [11], [26]). They generate the solutions with finite energy of the wave equation for elastic waves in an homogeneous isotropic half space with a free boundary.

In this article the main results concern the selfadjoint operator A_0. The operator A includes only the case of a free boundary condition and solid inclusions. More general examples including rigid and fluid inclusions will be considered in a next paper.

The results have been announced in [7] and [8]. But note that notations used in this article are different from these used in [7] and [8].

In a next paper distorded modes for A will be deduced from the limiting absorption principle and the scattering theory will be developped including the existence and the completness of the wave operators and representation theorems for the scattering matrix and the scattering amplitudes by using the distorded modes.

The article is organized as follows.

In the second section the selfadjoint operator A is defined and a qualitative description of the main results is given.

In the third section the spectral analysis of the selfadjoint operator A_0 is given. The set of the unperturbed modes appear as a complete set of generalized eigenfunctions of this selfadjoint operator. The limiting absorption principle in appropriate Hilbert spaces is given for the resolvent of A_0. Both of the half spaces $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2,\ x_2 > 0\}$ and $\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \in \mathbb{R}^3,\ x_3 > 0\}$ are considered.

In the fourth section we prove the basic division theorem for A_0.

Finally in the fifth section we give two different proofs of the limiting absorption principle in appropriate Hilbert spaces for the resolvent of A. We then deduce some spectral properties of A and give a result concerning decreasing properties of eigenfunctions of A associated with strictly positive eigenvalues.

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2. LINEAR ELASTICITY IN A PERTURBED ISOTROPIC HALF SPACE WITH A FREE BOUNDARY AND THE MAIN RESULT

Let $\mathbb{R}^3_+ = \{x = (x_1,x_2,x_3) \in \mathbb{R}^3; x_3>0\}$ and $|x|$ be the euclidian norm of x. Let L be a fixed strictly positive real number.

Ω denotes an open set of $\mathbb{R}^3$ such that $\Omega \cap \{x \in \mathbb{R}^3; |x| > L\} = \{x \in \mathbb{R}^3_+; |x| > L\}$. Let $\bar{\Omega}$ be the closure of Ω.

We assume that Ω satisfies the cone condition, i.e., there exist positive constants α, h such that for any $x \in \Omega$ one can construct a right spherical cone V_x with vertex x, opening α and height h such that it lies in Ω.

Ω can be the exterior in $\mathbb{R}^3_+$ of any compact set K whose interior is empty as for example a slit, a penny-shaped crack and, more generally, a compact flat crack of arbitrary shape such that the cone condition is satisfied.

We assume that an inhomogeneous isotropic solid occupies the domain Ω. The propagation of elastic waves is such a solid is governed by the following functions of $x \in \bar{\Omega}$

(2.1) $\rho(x)$ the equilibrium density of the solid

(2.2) $\lambda(x)$ and $\mu(x)$ the Lamé functions.

We assume that

(2.3) $0 < m \leq \lambda(x)$ (resp. $\mu(x)$, $\rho(x)$) $\leq M$ for a.e. x in $\bar{\Omega}$

and $\lambda(x) = \lambda_0$, $\mu(x) = \mu_0$ and $\rho(x) = \rho_0$ for $|x| > R$.

The state of the elastic field in the solid is determined by

(2.4) $u(x,t) = (u_1(x,t),u_2(x,t),u_3(x,t)) \in \mathbb{R}^3$, the displacement field of the solid at time t and position x

and

(2.5) $\sigma_{ij}(x,t)$, $i,j = 1,2,3$, the symmetric stress tensor field of the solid at time t and position x.

The equations satisfied by the elastic field in the isotropic solid are

$$\sigma_{ij}(u) = \lambda(.)(\nabla .u)\delta_{ij} + 2\mu(.)\, \varepsilon_{ij}(u) \tag{2.6}$$

where

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.7}$$

and

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{\rho(.)}\frac{\partial \sigma_{ij}}{\partial x_j}(u) \ . \tag{2.8}$$

Thus the displacement field u satisfies the following equations :

$$\frac{\partial^2 u_i}{\partial t^2} - \frac{1}{\rho(.)}\frac{\partial}{\partial x_j}(\lambda(.)(\nabla.u)\delta_{ij} + 2\mu(.)\varepsilon_{ij}(u)) = 0 \ . \tag{2.9}$$

The wave equation (2.9) must be supplemented by boundary conditions at the boundary $\partial\Omega$ of the solid. In the case of free boundaries the normal component of the stress must vanish at the boundary. Thus

$$\sigma_{ij}(u)\nu_j\big|_{\partial\Omega} = 0 \tag{2.10}$$

where ν is the exterior normal at point $x \in \partial\Omega$. We could consider rigid boundaries too, for which the displacement must vanish at the boundary.

In this article we restrict ourselves to free boundaries. It is one of the most important cases in pratical situations. Other cases will be dealt with along the same line in the next article.

With solutions to (2.9) and (2.10) with finite energy are usually associated a Hilbert space and a selfadjoint operator as follows.

Let

(2.11) $$(\mathcal{A}u)_i = -\frac{1}{\rho(.)}\frac{\partial}{\partial x_j}\sigma_{ij}(u) \quad \text{and}$$

$$L^2(\Omega,\mathcal{A},\mathbb{C}^3) = \{u \in L^2(\Omega,\mathbb{C}^3)\,;\mathcal{A}u \in L^2(\Omega,\mathbb{C}^3)\}$$

$H^m(\Omega,\mathbb{C}^3)$ denotes the usual Sobolev space. u in $H^1(\Omega,\mathbb{C}^3) \cap L^2(\Omega,\mathcal{A},\mathbb{C}^3)$ is said to satisfy the *generalized free boundary condition* if and only if one has

(2.12) $$\int_\Omega (\mathcal{A}u)_i \bar{v}_i \rho dx - \int_\Omega (\lambda(\nabla.u)(\nabla.\bar{v}) + 2\mu\, \varepsilon_{ij}(u)\varepsilon_{ij}(\bar{v}))dx = 0$$

for every v in $H^1(\Omega,\mathbb{C}^3)$.

The following operator $(D(A),A)$ in $L^2(\Omega,\mathbb{C}^3,\rho(x)dx)$:

(2.13) $D(A) = \{u \in H^1(\Omega,\mathbb{C}^3) \cap L^2(\Omega,\mathcal{A},\mathbb{C}^3); u$ satisfies the generalized free boundary condition$\}$

(2.14) $$Au = \mathcal{A}u \quad , \quad u \in D(A)$$

is a positive selfadjoint operator.

This is a consequence of the Korn's inequality (cf. [10], [15]) which follows from the fact that Ω satisfies the cone condition. The proof of the selfadjointness of A can be found in [19] or in [30]. If $\partial\Omega$ is smooth, then each u in $D(A)$ belongs to $H^2(\Omega,\mathbb{C}^3)$ (see [21], p. 222) and (2.10) is satisfied in the usual sense.

Let us consider a homogeneous isotropic half space $\mathbb{R}^3_+$ with a free boundary and with the density ρ_0 and the Lamé constants λ_0 and μ_0. We then define the positive selfadjoint operator $(D(A_0),A_0)$ in $L^2(\mathbb{R}^3_+,\mathbb{C}^3,\rho_0 dx)$ as follows (cf. [10], [19], [30], [21, p. 222])

(2.15)
$$D(A_0) = \{u \in H^2(\mathbb{R}^3_+,\mathbb{C}^3); \varepsilon_{13}(u)\big|_{x_3=0} = \varepsilon_{23}(u)\big|_{x_3=0} = \\ = \lambda_0 \nabla.u + 2\mu_0\varepsilon_{33}(u)\big|_{x_3=0} = 0\}.$$

(2.16)
$$A_0 u = -\frac{\lambda_0+\mu_0}{\rho_0}\nabla(\nabla.u) - \frac{\mu_0}{\rho_0}\Delta u \quad , \quad u \in D(A_0) ,$$

A is considered as a perturbation of A_0. We restrict ourselves to perturbations with compact support because it is always the case for applications. One could also consider short range perturbations of A_0 as in [6] and [29]. $A^{1/2}$ (resp. $A_0^{1/2}$) denotes the square root of A (resp. A_0). The solution of (2.9) and (2.10) satisfying the Cauchy conditions :

(2.17)
$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

is given by

(2.18)
$$u(x,t) = (\cos A^{1/2}t\ f)(x) + \left(\frac{\sin A^{1/2}t}{A^{1/2}}\ g\right)(x)$$

if f belongs to $D(A^{1/2})$ and g to $L^2(\Omega,\mathbb{C}^3,\rho(x)dx)$.

In fact $u(.,t)$ is in the class $C^1(\mathbb{R},L^2(\Omega,\mathbb{C}^3,\rho(x)dx)) \cap C^0(\mathbb{R},D(A^{1/2}))$ and satisfies the wave equation

$$\frac{d^2u}{dt^2} + Au = 0 \quad \text{for all} \quad t \in \mathbb{R} \tag{2.19}$$

and

$$u(0) = f \quad \text{and} \quad \frac{du}{dt}(0) = g \ . \tag{2.20}$$

In this case $u(.,t)$ is called a solution with finite energy (cf. [30]).

Let $E(K,u,t)$ be the restriction of the energy of u to a measurable subset K of Ω :

$$\begin{aligned} E(K,u,t) = \frac{1}{2} \sum_{i=1}^{3} \int_K \left|\frac{\partial u_i}{\partial t}\right|^2 \rho(x)dx + \sum_{i,j=1}^{3} \int_K (\lambda(x)|\nabla.u|^2 \\ + 2\mu(x)|\varepsilon_{ij}(u)|^2)dx \end{aligned} \tag{2.21}$$

$E(u,t)$ denotes the energy of u, i.e., $E(\mathbb{R}^3_+,u,t)$.

A study of further properties of u is based on a knowledge of the spectral properties of A.

It will be shown in section 3 that there exists a complete set of generalized eigenfunctions ψ_P, ψ_{SV}, ψ^0_{SV}, ψ_{SH} and ψ_R for A_0. Thus, in this case, the solution u, constructed from A_0, can be written as a sum of wave-packets of P-plane waves and their reflections, of S-plane waves and their reflections and of Rayleigh surface waves.

The spectrum of A_0 is $[0,\infty)$ and A_0 is an absolutely continuous operator (see [19] for definitions). This will be proved in section 3.

Every solution u with finite energy associated with A_0 is a transient state in the sense that, for every compact set K in $\mathbb{R}^3_+$, $E(K,u,t)$ tends to zero when t tends to $+\infty$.

The spectrum of A is $[0,\infty)$. Its structure is more complicated. Its continuous spectrum is also $[0,\infty)$ but A can have positive eigenvalues embedded in the continuous spectrum. Eigenvalues can accumulate only at 0 and $+\infty$. A has no continuous singular spectrum. All these results will be proved in section 5 and in the next article.

The main result of this article is the proof of the limiting absorption principle for A.

More precisely, let $R(z)$ (resp. $R_0(z)$) be the resolvent of A (resp. A_0) :

$$R(z) = (A-z)^{-1} \quad (\text{resp. } R_0(z) = (A_0-z)^{-1}) . \tag{2.22}$$

Then the limiting absorption principle states that $R_0(z)$ and $R(z)$ have a limit as bounded operators in appropriate Hilbert spaces when z tends to $\omega^2 > 0$ with a positive (or negative) imaginary part and when ω^2 is not an eigenvalue of A in the case of $R(z)$.

Rigourous statements will be given in sections 3 and 5.

3. THE SPECTRAL ANALYSIS OF A_0 IN $\mathbb{R}^3_+$ AND $\mathbb{R}^2_+$ AND THE LIMITING ABSORPTION PRINCIPLE

The spectral analysis of A_0 in $\mathbb{R}^2_+$ has been studied in [28]. J.R. Schulenberger has transformed the 2×2 second order system of linear elasticity into a 5×5 first order system. This approach introduces static solutions with no physical meaning. Here we consider the usual equations (see [1], [4], [5], [11]) and we follow the method developped in [15], [16], [17], [31] and [32] in order to analyse the spectral properties of A_0 and to construct a spectral representation for it. In this approach A_0 in $\mathbb{R}^3_+$ (resp. $\mathbb{R}^2_+$) is considered as unitarly equivalent to a direct integral of a field of 3×3 systems (resp. 2×2 systems) of Sturm-Liouville differential operators because of the symmetry of the problem. It is then sufficient to analyse the spectral properties of every 3×3 system (resp. 2×2 system) of Sturm-Liouville differential operators. We then construct a spectral representation for A_0 from which we deduce the limiting absorption principle. Some details are given for A_0 in $\mathbb{R}^3_+$. We only give the results for A_0 in $\mathbb{R}^2_+$ (see remark 3).

Let $p = (p_1, p_2) \in \mathbb{R}^2$ be the conjuguate variables of (x_1, x_2) and let $\mathcal{F}$ be the partial Fourier transform with respect to (x_1, x_2) :

$$(3.1) \qquad (\mathcal{F}u)(p,x_3) = \frac{1}{2\pi} \operatorname{L.i.m} \int_{\mathbb{R}^2} e^{-i(p_1x_1+p_2x_2)} u(x_1,x_2,x_3)dx_1dx_2$$

for u in $L^2(\mathbb{R}^3_+, \mathbb{C}^3, \rho_0 dx)$.

Let

$$(3.2) \qquad D(\hat{A}_0) = \mathcal{F}D(A_0)$$

$$(3.3) \qquad \hat{A}_0 = \mathcal{F}A_0\mathcal{F}^{-1}$$

$\hat{A}_0$ is a positive selfadjoint operator in $L^2(\mathbb{R}^3_+,\mathbb{C}^3,\rho_0 dx)$ unitarly equivalent to A_0.

Let

$$
\begin{aligned}
D(\hat{A}_0(p_1,p_2)) = \{u \in H^2(\mathbb{R}_+,\mathbb{C}^3);\ & \frac{du_1}{dx_3}(0) + ip_1u_3(0) = 0, \\
& \frac{du_2}{dx_3}(0) + ip_2u_3(0) = 0, \\
& (\lambda_0+2\mu_0)\frac{du_3}{dx_3}(0)+i\lambda_0p_1u_1(0) \\
& +i\lambda_0p_2u_2(0) = 0\}
\end{aligned}
\tag{3.4}
$$

and

$$
\hat{A}_0(p_1,p_2)\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}(x_3) = \tag{3.5}
$$

$$
\rho_0^{-1}\begin{pmatrix}
-\mu_0\frac{d^2}{dx_3^2}+(\lambda_0+2\mu_0)p_1^2+\mu_0p_2^2, & (\lambda_0+\mu_0)p_1p_2 & ,-ip_1(\lambda_0+\mu_0)\frac{d}{dx_3} \\
(\lambda_0+\mu_0)p_1p_2 & ,-\mu_0\frac{d^2}{dx_3^2}+\mu_0p_1^2+(\lambda_0+2\mu_0)p_2^2, & -ip_2(\lambda_0+\mu_0)\frac{d}{dx_3} \\
-ip_1(\lambda_0+\mu_0)\frac{d}{dx_3} & ,-ip_2(\lambda_0+\mu_0)\frac{d}{dx_3} & ,-(\lambda_0+2\mu_0)\frac{d^2}{dx_3^2}+\mu_0|p|^2
\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}(x_3)
$$

for u in $D(\hat{A}_0(p_1,p_2))$.

For each $p = (p_1,p_2) \in \mathbb{R}^2$, $\hat{A}_0(p_1,p_2)$ with its domain $D(\hat{A}_0(p_1,p_2))$ is a positive selfadjoint operator in $L^2(\mathbb{R}_+,\mathbb{C}^3,\rho_0 dy)$ (see [17]).

To get the spectral analysis of A_0, the first remark to be made is that $\hat{A}_0$ is the direct integral of the field $(\hat{A}_0(p_1,p_2))_{(p_1,p_2)\in\mathbb{R}^2}$ of selfadjoint operators :

$$\hat{A}_0 = \int_{\mathbb{R}^2}^{\oplus} \hat{A}_0(p_1,p_2)dp_1dp_2 \tag{3.6}$$

(see [15], [17], [32]).

The second remark to be made is that the isotropic half space is invariant with respect to any rotation around the x_3-axis. Using this property we show that for,every $(p_1,p_2) \neq 0$, $\hat{A}_0(p_1,p_2)$ is unitarily equivalent to a simpler operator whose study is easier.

For every $p \neq 0$ we consider the following 3×3 matrix :

$$U(p_1,p_2) = \frac{1}{|p|}\begin{pmatrix} p_1 & -p_2 & 0 \\ p_2 & p_1 & 0 \\ 0 & 0 & |p| \end{pmatrix} \tag{3.7}$$

where

$$|p| = (p_1^2 + p_2^2)^{1/2} . \tag{3.8}$$

We have

$$U(p_1,p_2)^{-1} = U(p_1,-p_2) \tag{3.9}$$

and

Proposition 3.1

For every $p = (p_1,p_2) \neq 0$ *one has*

$$\hat{A}_0(p_1,p_2) = U(p_1,p_2)\hat{A}_0(|p|,0)U(p_1,-p_2) \tag{3.10}$$

As it is shown in [17] $\hat{A}(|p|,0)$ is a direct sum of two selfadjoint operators respectively denoted by $B_1(|p|)$ and $B_2(|p|)$ and defined as follows :

$$\text{(3.11)}\qquad D(B_1(|p|)) = \left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^2(\mathbb{R}_+,\mathbb{C}^2); \frac{du_1}{dx_3}(0) + i|p|u_3(0) = (\lambda_0+2\mu_0)\frac{du_3}{dx_3}(0) + i|p|\lambda_0 u_1(0) = 0\right\}$$

and

$$\text{(3.12)}\qquad B_1(|p|)\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = \rho_0^{-1}\begin{bmatrix} -\mu_0\dfrac{d^2}{dx_3^2} + (\lambda_0+2\mu_0)|p|^2, & -i|p|(\lambda_0+\mu_0)\dfrac{d}{dx_3} \\ -i|p|(\lambda_0+\mu_0)\dfrac{d}{dx_3} & , -(\lambda_0+2\mu_0)\dfrac{d^2}{dx_3^2} + \mu_0|p|^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_3 \end{bmatrix}$$

$B_1(|p|)$ is a positive selfadjoint operator in $L^2(\mathbb{R}_+,\mathbb{C}^2,\rho_0 dx_3)$,

$$\text{(3.13)}\qquad D(B_2(|p|)) = \{u_2 \in H^2(\mathbb{R}_+);\ \frac{du_2}{dx_3}(0) = 0\}$$

$$\text{(3.14)}\qquad B_2(|p|)u_2 = -\mu_0\rho_0^{-1}\frac{d^2u_2}{dx_3^2} + \mu_0\rho_0^{-1}|p|^2 u_2$$

$B_2(|p|)$ is a positive selfadjoint operator in $L^2(\mathbb{R}_+,\mathbb{C},\rho_0 dx_3)$.

In fact we have

<u>Proposition 3.2</u>

For $p \neq 0$, $\hat{A}(|p|,0) = \Pi(B_1(|p|) \oplus B_2(|p|))\Pi$ *where* Π *is the following operator*

$$\text{(3.15)}\qquad \Pi\, {}^t(u_1,u_2,u_3) = {}^t(u_1,u_3,u_2)\ .$$

As well known $B_2(|p|)$ is an absolutely continuous operator whose spectrum is $[\mu_0/\rho_0 \ |p|^2,\infty)$. It is not the case for $B_1(|p|)$. In fact we have the following result.

Proposition 3.3

For every $p \neq 0$, $B_1(|p|)$ *has an unique simple eigenvalue* $c_R^2|p|^2$ *such that* $0 < c_R^2|p|^2 < (\mu_0/\rho_0)|p|^2$.
$c_R > 0$ *is the unique solution in* $(0,(\mu_0/\rho_0)^{1/2})$ *to the following implicit equation*

$$(3.16)\qquad (1-\frac{\alpha^2\rho_0}{2\mu_0})^2 - (1-\frac{\alpha^2\rho_0}{\mu_0})^{1/2}\,(1-\frac{\alpha^2\rho_0}{\lambda_0+2\mu_0})^{1/2} = 0\ .$$

For a proof of proposition 3.3 see [17]. This unique eigenvalue for $B_1(|p|)$ is the origin of the Rayleigh surface wave and c_R is its speed.

The spectral analysis of $B_1(|p|)$ and $B_2(|p|)$ is based on a formulation of the Weyl-Kodaira theory due to N. Dunford and J.T. Schwartz [9; chapter XIII]. However the theorems given in [9; chapter XIII] are not directely applicable to $B_1(|p|)$. One verifies that the Dunford-Schwartz theorems can be easily generalized to selfadjoint operators associated with 2×2 systems of Sturm-Liouville operators in order to get the spectral analysis and spectral representation of these selfadjoint realizations. In this article we only give the results.

Finally, from the spectral analysis of $B_1(|p|)$ and $B_2(|p|)$, from propositions 3.3, 3.2 and 3.1 and from (3.2) and (3.6) we obtain the spectral analysis of A_0. Expansions in terms of generalized eigenfunctions for A_0 are given. These generalized eigenfunctions are exactly the same as those used by physicists and engineers. They are P-plane waves and their reflections, S-plane waves and their reflections and the Rayleigh surface wave.

Let

$$c_P^2 = \frac{\lambda_0+2\mu_0}{\rho_0} \quad \text{and} \quad c_S^2 = \frac{\mu_0}{\rho_0} \tag{3.17}$$

$$E = \{(p,\omega) \in \mathbb{R}_+^3 \; ; \; \omega > c_P|p|\}$$

$$E_{SV}^0 = \{(p,\omega) \in \mathbb{R}_+^3 \; ; \; c_S|p| < \omega < c_P|p|\} \tag{3.18}$$

$$E_{SH} = \{(p,\omega) \in \mathbb{R}_+^3 \; ; \; c_S|p| < \omega\}$$

$$\begin{aligned} \xi_P(\omega^2) &= (\omega^2/c_P^2-|p|^2)^{1/2} \text{ for } (p,\omega) \in E \text{ and} \\ \xi_S(\omega^2) &= (\omega^2/c_S^2-|p|^2)^{1/2} \text{ for } (p,\omega) \in E_{SH} \end{aligned} \tag{3.19}$$

$$\xi_P'(\omega^2) = (|p|^2-\omega^2/c_P^2)^{1/2} \text{ for } \omega < c_P|p| \tag{3.20}$$

Definition 3.5

(i) For every $x \in \mathbb{R}_+^3$ *and for every* $(p,\omega) \in E$ *let*

$$\begin{aligned} \psi_P(x;p,\omega) = \frac{1}{(2\pi)^{3/2}} \frac{e^{i(p_1x_1+p_2x_2)}}{(\omega\rho_0\xi_P(\omega^2))^{1/2}} \Big[& e^{-i\xi_P(\omega^2)x_3} \; {}^t(p_1,p_2,-\xi_P(\omega^2)) \\ & + \frac{4|p|(\omega^2/c_S^2-2|p|^2)\xi_P(\omega^2)}{(\omega^2/c_S^2-2|p|^2)^2 + 4|p|^2\xi_P(\omega^2)\xi_S(\omega^2)} \\ & \quad e^{i\xi_S(\omega^2)x_3} \; {}^t\!\left(\frac{p_1}{|p|}\xi_S(\omega^2), \frac{p_2}{|p|}\xi_S(\omega^2), -|p|\right) \\ & - \frac{(\omega^2/c_S^2-2|p|^2)^2 - 4|p|^2\xi_P(\omega^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2|p|^2)^2 + 4|p|^2\xi_P(\omega^2)\xi_S(\omega^2)} \\ & \quad e^{i\xi_P(\omega^2)x_3} \; {}^t(p_1,p_2,\xi_P(\omega^2)) \Big] \end{aligned} \tag{3.21}$$

(ii) For every $x \in \mathbb{R}^3_+$ *and for every* $(p,\omega) \in E$ *let*

$$\psi_{SV}(x;p,\omega) = \frac{1}{(2\pi)^{3/2}} \frac{e^{i(p_1x_1+p_2x_2)}}{(\omega\rho_0\xi_S(\omega^2))^{1/2}} \Big[e^{-i\xi_S(\omega^2)x_3}\ {}^t(\tfrac{p_1}{|p|}\xi_S(\omega^2), \tfrac{p_2}{|p|}\xi_S(\omega^2), |p|)$$

$$+ \frac{(\omega^2/c_S^2-2|p|^2)^2-4|p|^2\xi_P(\omega^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2|p|^2)^2+4|p|^2\xi_P(\omega^2)\xi_S(\omega^2)}$$

$$e^{i\xi_S(\omega^2)x_3}\ {}^t(\tfrac{p_1}{|p|}\xi_S(\omega^2), \tfrac{p_2}{|p|}\xi_S(\omega^2), -|p|) \tag{3.22}$$

$$+ \frac{4|p|(\omega^2/c_S^2 - 2|p|^2)\xi_S(\omega^2)}{(\omega/c_S^2-2|p|^2)^2+4|p|^2\xi_P(\omega^2)\xi_S(\omega^2)}$$

$$e^{i\xi_P(\omega^2)x_3}\ {}^t(p_1, p_2, \xi_P(\omega^2))\Big]$$

(iii) For every $x \in \mathbb{R}^3_+$ *and for every* $(p,\omega) \in E^0_{SV}$ *let*

$$\psi^0_{SV}(x;p,\omega) = \frac{1}{(2\pi)^{3/2}} \frac{e^{i(p_1x_1+p_2x_2)}}{(\omega\rho_0\xi_S(\omega^2))^{1/2}}$$

$$\Big[e^{-i\xi_S(\omega^2)x_3}\ {}^t(\tfrac{p_1}{|p|}\xi_S(\omega^2), \tfrac{p_2}{|p|}\xi_S(\omega^2), |p|)$$

$$+ \frac{(\omega^2/c_S^2-2|p|^2)^2-4i|p|^2\xi'_P(\omega^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2|p|^2)^2+4i|p|^2\xi'_P(\omega^2)\xi_S(\omega^2)}$$

$$e^{i\xi_S(\omega^2)x_3}\ {}^t(\tfrac{p_1}{|p|}\xi_S(\omega^2), \tfrac{p_2}{|p|}\xi_S(\omega^2), -|p|)$$

(3.23) $$+ \frac{4|p|(\omega^2/c_S^2-2|p|^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2|p|^2)^2+4i|p|^2\xi_P'(\omega^2)\xi_S(\omega^2)} e^{-\xi_P'(\omega^2)x_3}\, {}^t(p_1,p_2,i\xi_P'(\omega^2))]$$

(iv) For every $x \in \mathbb{R}^3_+$ *and for every* $(p,\omega) \in E_{SH}$ *let*

(3.24) $$\psi_{SH}(x;p,\omega) = \frac{1}{c_S\pi^{3/2}}\left(\frac{\omega}{2\rho_0\xi_S(\omega^2)}\right)^{1/2} e^{i(p_1x_1,+p_2x_2)} \cos\xi_S(\omega^2)x_3 \; {}^t(-\frac{p_2}{|p|}, \frac{p_1}{|p|}, 0)$$

(v) For every $x \in \mathbb{R}^3_+$ *and for every* $p \in \mathbb{R}^2$ *let*

(3.25) $$\psi_R(x;p) = \frac{C}{2\pi} e^{i(p_1x_1+p_2x_2)} |p|^{1/2} [(2-(c_R/c_S)^2)e^{-|p|(1-(c_R/c_P)^2)^{1/2}x_3} \, {}^t(i\frac{p_1}{|p|}, i\frac{p_2}{|p|}, -(1-(c_R/c_P)^2)^{1/2}) - 2(1-(c_R/c_P)^2)^{1/2} e^{-|p|(1-(c_R/c_S)^2)^{1/2}x_3} \, {}^t(i\frac{p_1}{|p|}(1-(c_R/c_S)^2)^{1/2}, i\frac{p_2}{|p|}(1-(c_R/c_S^2)^2)^{1/2}, -1)]$$

C *is a strictly positive constant such that*

(3.26) $$4\pi^2 \int_0^\infty (|\psi_{R1}(x;p)|^2+|\psi_{R2}(x;p)|^2+|\psi_{R3}(x;p)|^2)\rho_0 dx_3 = 1 .$$

ψ_P, ψ_{SV}, ψ_{SV}^0, ψ_{SH} and ψ_R are generalized eigenfunctions for A_0.

We then have the following theorem

Theorem 3.6

For every f *in* $L^2(\mathbb{R}^3_+, \mathbb{C}^3, \rho_0 dx)$ *the following limits exist*

(3.27) $$\tilde{f}_P(p,\omega) = L^2(E)\text{-}\lim_{N\to\infty} \int_{\mathbb{R}^3_+ \cap \{|x|\le N\}} \overline{\psi_P(x;p,\omega)}.f(x)\rho_0 dx$$

(3.28) $$\tilde{f}_{SV}(p,\omega) = L^2(E)\text{-}\lim_{N\to\infty} \int_{\mathbb{R}^3_+ \cap \{|x|\le N\}} \overline{\psi_{SV}(x;p,\omega)}.f(x)\rho_0 dx$$

(3.29) $$\tilde{f}^0_{SV}(p,\omega) = L^2(E^0_{SV})\text{-}\lim_{N\to\infty} \int_{\mathbb{R}^3_+ \cap \{|x|\le N\}} \overline{\psi^0_{SV}(x;p,\omega)}.f(x)\rho_0 dx$$

(3.30) $$\tilde{f}_{SH}(p,\omega) = L^2(E_{SH})\text{-}\lim_{N\to\infty} \int_{\mathbb{R}^3_+ \cap \{|x|\le N\}} \overline{\psi_{SH}(x;p,\omega)}.f(x)\rho_0 dx$$

(3.31) $$\tilde{f}_R(p) = L^2(\mathbb{R}^2)\text{-}\lim_{N\to\infty} \int_{\mathbb{R}^3_+ \cap \{|x|\le N\}} \overline{\psi_R(x;p)}.f(x)\rho_0 dx$$

where a.b *is the usual scalar product in* $\mathbb{R}^3$ *and*

$$\overline{\psi(x;p,\omega)} = {}^t(\overline{\psi_1(x;p,\omega)}, \overline{\psi_2(x;p,\omega)}, \overline{\psi_3(x;p,\omega)}).$$

Let

(3.32) $$\Phi_P f = \tilde{f}_P,\ \Phi_{SV} f = \tilde{f}_{SV},\ \Phi^0_{SV} f = \tilde{f}^0_{SV},\ \Phi_{SH} f = \tilde{f}_{SH},\ \Phi_R f = \tilde{f}_R .$$

Each operator $\Phi.$ *is a partial isometric one such that*

(3.33) $$\Phi.\Phi^*_. = I$$

Moreover the following limits exist in $L^2(\mathbb{R}^3_+, \mathbb{C}^3, \rho_0 dx)$:

(3.34) $$(\Phi^*_P \tilde{f}_P)(x) = \text{L.i.m.} \int_E \psi_P(x;p,\omega)\tilde{f}_P(p,\omega)\,dp\,d\omega$$

(3.35) $$(\Phi^*_{SV} \tilde{f}_{SV})(x) = \text{L.i.m.} \int_E \psi_{SV}(x;p,\omega)\tilde{f}_{SV}(p,\omega)\,dp\,d\omega$$

(3.36) $$(\Phi_{SV}^{*0}\tilde{f}_{SV}^{0})(x)=\text{L.i.m.}\int_{E_{SV}^{0}} \psi_{SV}^{0}(x;p,\omega)\tilde{f}_{SV}^{0}(p,\omega)dpd\omega$$

(3.37) $$(\Phi_{SH}^{*}\tilde{f}_{SH})(x)=\text{L.i.m.}\int_{E_{SH}} \psi_{SH}(x;p,\omega)\tilde{f}_{SH}(p,\omega)dpd\omega$$

(3.38) $$(\Phi_{R}^{*}\tilde{f}_{R})(x)=\text{L.i.m.}\int_{\mathbb{R}^2} \psi_{R}(x;p)\tilde{f}_{R}(p)dp\ .$$

Let Φ *be the following application*

(3.39) $$\Phi f = (\Phi_P f, \Phi_{SV} f, \Phi_{SV}^{0} f, \Phi_{SH} f, \Phi_R f)$$

Φ *is an unitary operator from* $L^2(\mathbb{R}_+^3, \mathbb{C}^3, \rho_0 dx)$ *into*

(3.40) $$L^2(E) \oplus L^2(E) \oplus L^2(E_{SV}^{0}) \oplus L^2(E_{SH}) \oplus L^2(\mathbb{R}^2)\ .$$

Furthermore f *belongs to* $D(A_0)$ *if and only if*

(3.41) $$\int_E \omega^4(|\tilde{f}_P(p,\omega)|^2+|\tilde{f}_{SV}(p,\omega)|^2)dpd\omega + \int_{E_{SV}^{0}} \omega^4|\tilde{f}_{SV}^{0}(p,\omega)|^2 dpd\omega + \int_{E_{SH}} \omega^4|\tilde{f}_{SH}(p,\omega)|^2 dpd\omega + \int_{\mathbb{R}^2} |p|^4|\tilde{f}_R(p)|^2 dp < \infty$$

and

(3.42) $$\Phi(A_0 f) = (\omega^2\tilde{f}_P, \omega^2\tilde{f}_{SV}, \omega^2\tilde{f}_{SV}^{0}, \omega^2\tilde{f}_{SH}, c_R^2|p|^2\tilde{f}_R)$$

if $f \in D(A_0)$.

In particular A_0 *is an absolutely continuous selfadjoint operator whose spectrum is* $[0,\infty)$.

The limiting absorption principle for A_0 is a consequence of Theorem 3.6. The proof will be exactly the same as for the stratified media (cf. [6], [29]). Thus we only give the main results for A_0.

Let s_1 and s_2 be two real numbers. We denote $L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ the space of all measurable $\mathbb{C}^3$ valued functions on $\mathbb{R}^3_+$ defined by

$$(3.43)\qquad L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3) = \{u(x):(1+x_1^2+x_2^2)^{s_1/2}(1+x_3^2)^{s_2/2}\,u(x)\in L^2(\mathbb{R}^3_+,\mathbb{C}^3)\}\ .$$

In $L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ we introduce the norm

$$(3.44)\qquad ||u||^2_{0;s_1,s_2} = \int_{\mathbb{R}^3_+}(1+x_1^2+x_2^2)^{s_1}(1+x_3^2)^{s_2}\,u(x).\overline{u(x)}dx\ .$$

More generally we consider weighted Sobolev L^2 spaces $H^{m;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ defined for any integer $m \geq 0$ by

$$(3.45)\qquad H^{m;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3) = \{u(x);D^\alpha u\in L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)\ ,\ 0\leq|\alpha|\leq m\}$$

$H^{m;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ is a Hilbert space under the norm :

$$(3.46)\qquad ||u||^2_{m;s_1,s_2} = \sum_{|\alpha|\leq m}||D^\alpha u||^2_{0;s_1,s_2}\ .$$

We then have the following theorem :

<u>Theorem 3.7</u> (Limiting absorption principle for A_0)

Suppose $s_1 > \frac{1}{2}$ *and* $s_2 > \frac{1}{2}$; $z \to R_0(z)$ *is an analytic operator valued function on* $\mathbb{C}\setminus[0,\infty)$ *with values in* $B(L^{2;s_1,s_2},H^{2;-s_1,-s_2})$. *Then*

(i) For any $\omega_0 > 0$ *the following two limits exist in the uniform operator topology of* $B(L^{2;s_1,s_2},H^{2;-s_1,-s_2})$:

$$(3.47)\qquad R_0^\pm(\omega_0^2) = \lim_{\substack{z\to\omega^2\\ \pm\mathrm{Im}z>0}} R_0(z)\ .$$

Moreover, for any $f\in L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ *the function* $u = R_0^\pm(\omega_0^2)f$ *verifies the differential equation*

(3.48) $$(c_P^2-c_S^2)\nabla(\nabla.u) + c_S^2\Delta u + \omega_0^2 u = 0$$

(ii) For any compact interval $[a,b]$ *contained in* $(0,\infty)$ *let*

(3.49) $$J^{\pm}(a,b)=\{z \in \mathbb{C};\mathrm{Re}z \in [a,b] \text{ and } \pm\mathrm{Im}z \in [0,1]\}$$

$$R_0^{\pm}(z) = R(z) \quad \text{if} \quad \mathrm{Im}z \neq 0 \quad \text{and} \quad z \in J^{\pm}(a,b)$$

For every δ *such that* $\delta < \inf(1,s_1-1/2,s_2-1/2)$ *there exists a positive constant* C *depending on* A_0,a,b *and* δ *such that*

(3.50) $$||R_0^+(z_1)-R_0^+(z_2)||_{B(L^{2;s_1,s_2},H^{2;-s_1,-s_2})} \le C|z_1-z_2|^{\delta}$$

(3.51) $$(\textit{resp.}\ ||R_0^-(z_1)-R_0^-(z_2)||_{B(L^{2;s_1,s_2},H^{2;-s_1,-s_2})} \le C|z_1-z_2|^{\delta})$$

for every $(z_1,z_2) \in J^+(a,b) \times J^+(a,b)$ (resp. $J^-(a,b) \times J^-(a,b)$).

As in [6] in order to prove the limiting absorption principle we have to define generalized trace operators associated with A_0. For any $\omega > 0$ let

(3.52)
$$E(\omega) = \{p \in \mathbb{R}^2;(p,\omega) \in E\}$$
$$E_{SV}^0(\omega) = \{p \in \mathbb{R}^2;(p,\omega) \in E_{SV}^0\}$$
$$E_{SH}(\omega) = \{p \in \mathbb{R}^2;(p,\omega) \in E_{SH}\}$$
$$E_R(\omega) = \{p \in \mathbb{R}^2;c_R|p| = \omega\}.$$

We then have

Proposition 3.8

Suppose $s_1 > 1/2$ *and* $s_2 > 1/2$. *For any* $\omega > 0$ *there exist generalized trace operators* $\tau_P(\omega)$, $\tau_{SV}(\omega)$, $\tau^0_{SV}(\omega)$, $\tau_{SH}(\omega)$ *and* $\tau_R(\omega)$ *from* $L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ *into, respectively,* $L^2(E(\omega))$, $L^2(E(\omega))$, $L^2(E^0_{SV}(\omega))$, $L^2(E_{SH}(\omega))$, *and* $L^2(E_R(\omega))$, *such that for any* $f \in C^\infty_0(\mathbb{R}^3_+,\mathbb{C}^3)$:

$$(\tau_P(\omega)f)(p) = \tilde{f}_P(p,\omega)$$

$$(\tau_{SV}(\omega)f)(p) = \tilde{f}_{SV}(p,\omega)$$

$$(\tau^0_{SV}(\omega)f)(p) = \tilde{f}^0_{SV}(p,\omega) \tag{3.53}$$

$$(\tau_{SH}(\omega)f)(p) = \tilde{f}_{SH}(p,\omega)$$

$$(\tau_R(\omega)f)(p) = \tilde{f}_R(p) \ .$$

Furthermore for any $f \in L^2(\mathbb{R}^3_+,\mathbb{C}^3)$ *one has*

$$||\tau_P(\omega)f||_{L^2(E(\omega))} \le M(\omega)\ ||f||_{0;s_1,s_2} \tag{3.54}$$

(resp. $||\tau_{SV}(\omega)f||$, $||\tau^0_{SV}(\omega)f||$, $||\tau_{SH}(\omega)f||$, $||\tau_R(\omega)f||$ *)* .

where $M(\omega)$ *is a continuous function on* $(0,\infty)$.

The proof of proposition 3.8 is exactly as in [6].

Now, for f and g in $C^\infty_0(\mathbb{R}^3_+)$, we have

$$\int_{\mathbb{R}^3_+} R_0(z)f.g\rho_0 dx = B_P(z,f,g) + B_{SV}(z,f,g) + B^0_{SV}(z,f,g) + \\ + B_{SH}(z,f,g) + B_R(z,f,g) \tag{3.55}$$

where

$$(3.56) \qquad B_P(z,f,g) = \int_0^\infty h_P(\omega,f,g)\,\frac{d\omega}{\omega^2-z}$$

(resp. $B_{SV}(z,f,g)$, ...)

with

$$(3.57) \qquad h_P(\omega,f,g) = \int_{E(\omega)} \tilde{f}_P(p,\omega)\,\overline{\tilde{g}_P(p,\omega)}\,d\sigma_P(\omega)$$

(resp. $h_{SV}(\omega,f,g)$, ...)

where $d\sigma_P(.)$ is the measure on $E(\omega)$ induced by the Lebesgue one.

Proposition 3.8 gives a meaning to (3.56) for any f and g in $L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$.

Useful properties of $h_P(.,.,.)$ are now described in the following lemma.

Lemma 3.9

Suppose $s_1 > 1/2$, $s_2 > 1/2$, $\delta < \inf(s_1-1/2, s_2-1/2)$ *and* $\delta \in [0,1]$ *Then there exist two continuous positive functions* $M(.)$ *and* $M(.,.)$ *respectively on* $(0,\infty)$ *and* $(0,\infty)\times(0,\infty)$ *such that*

$$(3.58) \qquad |h_P(\omega,f,g)| \le M(\omega)\,||f||_{0;s_1,s_2}\,||g||_{0;s_1,s_2}$$

(resp. $|h_{SV}(\omega,f,g)|$, $|h^0_{SV}(\omega,f,g)|$, $|h_{SH}(\omega,f,g)|$, $|h_R(\omega,f,g)|$ *)*

$$(3.59) \qquad |h_P(\omega,f,g)-h_P(\omega',f,g)| \le M(\omega,\omega')\,|\omega-\omega'|^\delta\,||f||_{0;s_1,s_2}\,||g||_{0;s_1,s_2}$$

(resp. $|h_{SV}(\omega,f,g)-h_{SV}(\omega',f,g)|$, ...) *)*.

It follows from the lemma 3.9 that $B_P(\omega_0^2 \pm i\varepsilon,f,g)$ (resp. $B_{SV}(\omega_0^2 \pm i\varepsilon,f,g)$,...) tends to a limit denoted by $B_P(\omega_0^2 \pm i0,f,g)$ when ε tends to 0_+. We have

$$(3.60) \qquad B_P(\omega_0^2 \pm i0, f, g) = PV \int_E \frac{\tilde{f}_P(p,\omega)\overline{\tilde{g}_P(p,\omega)}}{\omega^2 - \omega_0^2} \, dp d\omega \pm i \frac{\pi}{2\omega_0} h_P(\omega_0, f, g)$$

$$(\text{resp. } B_{SV}(\omega_0^2 \pm io, f, g), \ \ldots\ ) .$$

From the above results and from the proof of theorem 4.1 of [3] we immediately deduce (i) of theorem 3.7. Then (ii) is a consequence of (3.56), (3.57) and (3.59). Thus theorem 3.7 is proved.

Finally we conclude this section with three remarks.

Remark 1

As for the remarks 2.7 and 2.8 in [6] we have the following lemma

Lemma 3.10

Suppose $s_1 > 1/2$ *and* $s_2 > 1/2$. *Consider* $\omega_0 > 0$ *and* $f \in L^{2;s_1,s_2}(\mathbb{R}^3_+, \mathbb{C}^3)$. *Then the following statements are equivalent :*

(3.61) *(i)* $\qquad R_0^+(\omega_0^2)f = R_0^-(\omega_0^2)f$

(3.62) *(ii)* $\qquad \mathrm{Im} \int_{\mathbb{R}^3_+} R_0^+(\omega_0^2)f.\bar{f}\rho_0 dx = 0$

(3.63) *(iii)* $\qquad \mathrm{Im} \int_{\mathbb{R}^3_+} R_0^-(\omega_0^2)f.\bar{f}\rho_0 dx = 0$

(3.64) *(iv)* $\qquad \tau_P(\omega_0)f = \tau_{SV}(\omega_0)f = \tau_{SV}^0(\omega_0)f = \tau_{SH}(\omega_0)f = \tau_R(\omega_0)f = 0$

(3.65) *(v)* $\qquad \tau_P(\omega_0)\bar{f} = \tau_{SV}(\omega_0)\bar{f} = \tau_{SV}^0(\omega_0)\bar{f} = \tau_{SH}(\omega_0)\bar{f} = \tau_R(\omega_0)\bar{f} = 0$.

Remark 2

Consider the following Cauchy problem :

$$(3.66) \qquad \frac{d^2u}{dt^2} + A_0 u = 0$$

(3.67) $u(0) = f \qquad \frac{du}{dt}(0) = g$

with $f \in D(A_0^{1/2})$ and $g \in L^2(\mathbb{R}^3_+, \mathbb{C}^3, \rho_0 dx)$.

Theorem 3.6 implies the following representation of the solution $u(t)$ of (3.66) and (3.67)

$$u(t,x) = \text{L.i.m.}\int_E \psi_P(x;p,\omega)\,[\cos\omega t\ \tilde f_P(p,\omega) + \frac{\sin\omega t}{\omega}\ \tilde g_P(p,\omega)]dp d\omega$$

$$+\text{L.i.m.}\int_E \psi_{SV}(x;p,\omega)[\cos\omega t\ \tilde f_{SV}(p,\omega) + \frac{\sin\omega t}{\omega}\ \tilde g_{SV}(p,\omega)]dp d\omega$$

$$+\text{L.i.m.}\int_{E^0_{SV}} \psi^0_{SV}(x;p,\omega)[\cos\omega t\ \tilde f^0_{SV}(p,\omega) + \frac{\sin\omega t}{\omega}\ \tilde g^0_{SV}(p,\omega)]dp d\omega \tag{3.68}$$

$$+\text{L.i.m.}\int_{E_{SH}} \psi_{SH}(x;p,\omega)[\cos\omega t\ \tilde f_{SH}(p,\omega) + \frac{\sin\omega t}{\omega}\ \tilde g_{SH}(p,\omega)]dp d\omega$$

$$+\text{L.i.m.}\int_{\mathbb{R}^2} \psi_R(x;p)[\cos c_R|p|t\ \tilde f_R(p) + \frac{\sin c_R|p|t}{c_R|p|}\ \tilde g_R(p)]dp\ .$$

From (3.68) it is not difficult to deduce that $u(t,.)$ behaves when t tends to $+\infty$ as a sum of an outgoing P-wave, propagating with velocity c_P, two outgoing S-waves propagating with velocity c_S and an outgoing cylindrical Rayleigh surface wave propagating with velocity c_R. This is a consequence of the stationary phase method as in [33].

Remark 3

For the half space $\mathbb{R}^2_+$ the above results are exactly the same as for $\mathbb{R}^3_+$. The only difference is that for $n = 2$ the operator A_0 has only four generalized eigenfunctions denoted by $\psi_P(x,p,\omega)$, $\psi_S(x,p,\omega)$ $\psi^0_S(x,p,\omega)$ and $\psi_R(x,p)$ where $x = (x_1,x_2) \in \mathbb{R}^2_+$ and $p \in \mathbb{R}$. This can be proved by following the same method as for $n = 3$.

Let $\xi_P(\omega^2) = (\omega^2/c_P^2-p^2)^{1/2}$ if $\omega > c_P|p|$, $\xi_P'(\omega^2) = (p^2-\omega^2/c_P^2)^{1/2}$ if $c_S|p| < \omega < c_P|p|$ and $\xi_S(\omega^2) = (\omega^2/c_S^2-p^2)^{1/2}$ if $\omega > c_S|p|$.

Let

$$E = \{(p,\omega) \in \mathbb{R}_+^2 \ ; \ \omega > c_P|p| \ \}$$

(3.69)

$$E_S^0 = \{(p,\omega) \in \mathbb{R}_+^2 \ ; \ c_S|p| < \omega < c_P|p|\}.$$

For any $(p,\omega) \in E$ we have

$$\psi_P(x,p,\omega) = \frac{1}{2\pi} e^{ipx_1} \frac{1}{(\omega\rho_0\xi_P(\omega^2))^{1/2}} \{e^{-i\xi_P(\omega^2)x_2} \ {}^t(p,-\xi_P(\omega^2))$$

(3.70)

$$+ \frac{4p(\omega^2/c_S-2p^2)\xi_P(\omega^2)}{(\omega^2/c_S^2-2p^2)^2+4p^2\xi_P(\omega^2)\xi_S(\omega^2)} e^{i\xi_S(\omega^2)x_2} \ {}^t(\xi_S(\omega^2),-p)$$

$$- \frac{(\omega^2/c_S^2-2p^2)^2-4p^2\xi_P(\omega^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2p^2)^2+4p^2\xi_P(\omega^2)\xi_S(\omega^2)} e^{i\xi_P(\omega^2)x_2} \ {}^t(p,\xi_P(\omega^2))\}$$

$$\psi_S(x,p,\omega) = \frac{1}{2\pi} e^{ipx_1} \frac{1}{(\omega\rho_0\xi_S(\omega^2))^{1/2}} \{e^{-i\xi_S(\omega^2)x_2} \ {}^t(\xi_S(\omega^2),p)$$

(3.71)

$$+ \frac{(\omega^2/c_S^2-2p^2)^2-4p^2\xi_P(\omega^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2p^2)^2+4p^2\xi_P(\omega^2)\xi_S(\omega^2)} e^{i\xi_S(\omega)x_2} \ {}^t(\xi_S(\omega^2),-p)$$

$$+ \frac{4p(\omega^2/c_S^2-2p^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2p^2)^2+4p^2\xi_P(\omega^2)\xi_S(\omega^2)} e^{i\xi_P(\omega^2)x_2} \ {}^t(p,\xi_P(\omega^2))\} .$$

For any $(p,\omega) \in E_S^0$ we have

$$\psi_S^0(x,p,\omega) = \frac{1}{2\pi} e^{ipx_1} \frac{1}{(\omega\rho_0\xi_S(\omega^2))^{1/2}} \{e^{-i\xi_S(\omega^2)x_2}\ {}^t(\xi_S(\omega^2),p)$$

$$(3.72) \qquad + \frac{(\omega^2/c_S^2-2p^2)^2-4ip^2\xi_P'(\omega^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2p^2)^2+4ip^2\xi_P'(\omega^2)\xi_S(\omega^2)} e^{i\xi_S(\omega^2)x_2}\ {}^t(\xi_S(\omega^2),-p)$$

$$+ \frac{4p(\omega^2/c_S^2-2p^2)\xi_S(\omega^2)}{(\omega^2/c_S^2-2p^2)^2+4ip^2\xi_P'(\omega^2)\xi_S(\omega^2)} e^{-\xi_P'(\omega^2)x_2}\ {}^t(p,i\xi_P'(\omega^2))\}$$

and for any $p \in \mathbb{R}$ we have

$$\psi_R(x,p) = \frac{C}{2\pi} e^{ipx_1} |p|^{1/2}\{(2-\frac{c_R^2}{c_S^2}) e^{-|p|(1-c_R^2/c_P^2)^{1/2}x_2}$$

$$(3.73) \qquad {}^t(i,-(1-c_R^2/c_P^2)^{1/2}\frac{p}{|p|})$$

$$-2(1-\frac{c_R^2}{c_P^2})^{1/2} e^{-|p|(1-c_R^2/c_S^2)^{1/2}x_2}\ {}^t(i(1-c_R^2/c_S^2)^{1/2},-\frac{p}{|p|})\}$$

where C is a positive constant such that

$$(3.74) \qquad 4\pi^2 \int_0^\infty |\psi_R(x,p)|^2\rho_0 dx_2 = 1 .$$

4. THE DIVISION THEOREM FOR A_0

This section is devoted to the generalization for A_0 of theorem B.1 proved by S. Agmon in [3] (see also [18], [25], [27]). This is the so called division theorem for A_0. Roughly speaking, this theorem says that if the generalized traces of $f \in L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ vanish on $E(\omega)$, $E(\omega)$, $E^0_{SV}(\omega)$, $E_{SH}(\omega)$ and $E_R(\omega)$ then the function $u = R^{\pm}_0(\omega^2)f$ has a better decay at infinity than it is expected from theorem 3.7.

As a consequence of the division theorem we deduce the limiting absorption principle for A without using any explicit radiation condition and any subsequent uniqueness theorem as the Rellich's theorem for the Laplacian in $\mathbb{R}^n$. Moreover the division theorem enables us to prove decay properties on the x_3-direction of the eigenfunctions for A associated with the strictly positive eigenvalues.

Thus we have

<u>Theorem 4.1</u> (the division theorem for A_0)

Assume $s_1 > 1/2$ *and* $s_2 > 1/2$. *Let* $f \in L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ *and let* σ *be a strictly positive number such that*

(4.1) $$\tau_P(\sigma)f = \tau_{SV}(\sigma)f = \tau^0_{SV}(\sigma)f = \tau_{SH}(\sigma)f = \tau_R(\sigma)f = 0 .$$

Let $\tilde{s}_1$ *and* $\tilde{s}_2$ *be two real numbers such that*

(4.2) $$\tilde{s}_1 = \begin{cases} s_1-1 & \text{if } 1/2 < s_1 \le 1 \\ 0 & \text{if } 1 < s_1 \end{cases}$$

and

(4.3) $$\begin{aligned} &\tilde{s}_2 < \inf(2s_1-3/2, s_2-1) \quad \text{if} \quad 1/2 < s_2 \le 1 \\ &\tilde{s}_2 < \inf(2s_1-3/2, s_2/2-1/2, 1) \quad \text{if} \quad 1 < s_2 . \end{aligned}$$

Then

$$(4.4) \qquad R_0^+(\sigma^2)f = R_0^-(\sigma^2)f \in L^{2;\tilde{s}_1,\tilde{s}_2}(\mathbb{R}_+^3,\mathbb{C}^3)$$

and

$$(4.5) \qquad ||R_0^+(\sigma^2)f||_{0;\tilde{s}_1,\tilde{s}_2} \le M(\sigma)\ ||f||_{0;s_1,s_2}$$

where $M(.)$ *is a positive continuous function on* $(0,\infty)$ *depending only on* s_1, s_2 *and* $\tilde{s}_2$.

Remark

Continuity of $M(.)$ is useful in proving that the positive eigenvalues of A can accumulate only at 0 and $+\infty$.

The proof of theorem 4.1 is very long and depends on several propositions.

We have (cf. section 3) :

$$(4.6) \qquad A_0 = \mathcal{F}^{-1}\hat{A}_0\mathcal{F} = \mathcal{F}^{-1}(\int^{\oplus}_{p\in\mathbb{R}^2} U(p_1,p_2)\Pi(B_1(|p|)\oplus B_2(|p|))\Pi U(p_1,-p_2)dp)\mathcal{F} .$$

For any $z \in \mathbb{C} \setminus [0,\infty)$ let

$$R_1(z) = (B_1(|p|)-zI)^{-1} \text{ and } R_2(z) = (B_2(|p|)-zI)^{-1} .$$

We then have

$$(4.7) \qquad R_0(z) = \mathcal{F}^{-1}(\int^{\oplus}_{p\in\mathbb{R}^2} U(p_1,p_2)\Pi(R_1(z)\oplus R_2(z))\Pi U(p_1,-p_2)dp)\mathcal{F} .$$

Assume $s_1 > 1/2$, $s_2 > 1/2$ and $f \in L^2(\mathbb{R}_+^3,\mathbb{C}^3)$. Let

$$(4.8) \qquad g(p,x_3) = {}^t(g_1(p,x_3),g_2(p,x_3),g_3(p,x_3)) = U(p_1,-p_2)(\mathcal{F}f)(p,x_3)$$

and

(4.9) $\quad g'(p,x_3) = (g_1(p,x_3),\ g_3(p,x_3))$.

Thus $g \in L^{2;0,s_2}(\mathbb{R}^3_+,\mathbb{C}^3,dpdx_3)$.

The method used in section 3 gives a meaning to $R_1^{\pm}(\sigma^2)g'$ and to $R_2^{\pm}(\sigma^2)g_2$. Thus we have

(4.10) $\quad (\mathscr{F}R_0^{\pm}(\sigma^2)f)(p,x_3) = U(p_1,p_2)\Pi(R_1^{\pm}(\sigma^2)g'(p,.)\oplus R_2^{\pm}(\sigma^2)g_2(p,.))(x_3)$.

Thus the proof of theorem 4.1 is the consequence of (4.10) and of a careful study of $R_1^{\pm}(\sigma^2)g'(p)$ and $R_2^{\pm}(\sigma^2)g_2(p)$.

First of all we have to give definitions of several analytic functions in the cut complex plane

(i) $\xi_S(z)$ denotes the branch of $(z/c_S^2-|p|^2)^{1/2}$ on $\mathbb{C}\backslash[-\infty,c_S^2|p|^2]$ such that $\mathrm{Re}\xi_S(z) > 0$.

$\tilde{\xi}_S(z)$ denotes the branch of $(z/c_S^2-|p|^2)^{1/2}$ on $\mathbb{C}\backslash[c_S^2|p|^2,\infty)$ such that $\mathrm{Im}\tilde{\xi}_S(z) > 0$.

$\xi_S'(z)$ denotes the branch of $(|p|^2-z/c_S^2)^{1/2}$ on $\mathbb{C}\backslash[c_S^2|p|^2,\infty)$ such that $\mathrm{Re}\xi_S'(z) > 0$.

(ii) $\xi_P(z)$ denotes the branch of $(z/c_P^2-|p|^2)^{1/2}$ on $\mathbb{C}\backslash(-\infty,c_P^2|p|^2]$ such that $\mathrm{Re}\xi_P(z) > 0$.

$\tilde{\xi}_P(z)$ denotes the branch of $(z/c_P^2-|p|^2)^{1/2}$ on $\mathbb{C}\backslash[c_P^2|p|^2,\infty)$ such that $\mathrm{Im}\tilde{\xi}_P(z) > 0$.

$\xi_P'(z)$ denotes the branch of $(|p|^2-z/c_P^2)^{1/2}$ on $\mathbb{C}\backslash[c_P^2|p|^2,\infty)$ such that $\mathrm{Re}\xi_P'(z) > 0$.

One verifies that

(4.11) $\quad \mathrm{Im}\xi_S(z)\mathrm{Im}z > 0$ and $\mathrm{Im}\xi_S'(z)\mathrm{Im}z < 0$ if $\mathrm{Im}z \neq 0$.

Moreover one has

(4.12) $\xi_S(z) = \tilde{\xi}_S(z) = i\xi'_S(z)$ and $\xi_P(z) = \tilde{\xi}_P(z) = i\xi'_P(z)$ if $\operatorname{Im} z > 0$

and

(4.13) $\xi_S(z) = -\tilde{\xi}_S(z) = -i\xi'_S(z)$ and $\xi_P(z) = -\tilde{\xi}_P(z) = -i\xi'_P(z)$ if $\operatorname{Im} z < 0$.

4.1. Study of $R_2^{\pm}(\sigma^2)g_2$

Let

(4.14) $v(p,x_3,\sigma^2) = (R_2^+(\sigma^2)g_3(p,.))(x_3)$

We then have

Proposition 4.2

Suppose $s_1 > 1/2$ *and* $s_2 > 1/2$. *Let* $f \in L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ *and let* σ *be a strictly positive number such that*

(4.15) $\tau_{SH}(\sigma)f = 0$.

Let δ *be a real number such that*

(4.16) $\delta < \inf(2s_1-1, s_2-1/2)$ *if* $1/2 < s_2 \le 1$

(4.17) $\delta < \inf(2s_1-1, s_2/2, 3/2)$ *if* $1 < s_2$.

Then

$$v(.,.,\sigma^2) = R_2^+(\sigma^2)g_2 = R_2^-(\sigma^2)g_2 \in L^2(\mathbb{R}^3_+,(1+x_3^2)^{\delta-1/2}dp\,dx_3)$$

and

(4.18) $$\|v(.,.,\sigma^2)\|_{L^2(\mathbb{R}^3_+,(1+x_3^2)^{\delta-1/2}dp\,dx_3)} \le M_\delta(\sigma)\,\|f\|_{0;s_1,s_2}$$

where $M_\delta(.)$ *is a positive continuous function on* $(0,\infty)$.

(4.15) is equivalent to

$$(4.19)\qquad \int_0^\infty \cos\xi_S(\sigma^2)x_3\; g_2(p,x_3)dx_3 = 0 \quad \text{for} \quad |p| < \sigma/c_S \;.$$

Let χ_1 (resp. χ_2,χ_3) be the characteristic function of $(0,\sigma/c_S)$ (resp. $(\sigma/c_S,\ \sqrt{2}\ \sigma/c_S)$, $(\sqrt{2}\ \sigma/c_S,\infty)$).

Let

$$(4.20)\qquad v_i(p,x_3,z) = \chi_i(|p|)v(p,x_3,z) = \chi_i(|p|)[R_2(z)g_2(p,.)](x_3)$$
$$i = 1,2,3 \;.$$

$v_i(p,x_3,z)$ has a meaning for $z \in \mathbb{C}\backslash[0,\infty)$. $v_i(p,x_3,\sigma^2)$ will be defined as the limit of $v_2(p,x_3,z)$ as z tends to σ^2 such that $\text{Im}z > 0$.

After having estimated each $v_i(p,x_3,\sigma^2)$ it will not be difficult to show as in [6] that

$$(4.21)\qquad v(p,x_3,\sigma^2) = \sum_{i=1}^{3} v_i(p,x_3,\sigma^2) \;.$$

4.1.1. Definition and estimate of $v_1(p,x_3,\sigma^2)$

Let

$$\phi_1(p,x_3,z) = \cos\,\xi_S(z)x_3 \quad \text{and} \quad \phi_2(p,x_3,z) = \exp\, i\xi_S(z)x_3 \;.$$

It follows from theorem XIII.3.16 in [9] that

$$(4.22)\qquad \begin{aligned} v_1(p,x_3,z) = {} & i\ \chi_1(|p|)(c_S^2\xi_S(z))^{-1}[\phi_2(p,x_3,z)\int_0^{x_3}\phi_1(p,y,z)g_2(p,y)dy \\ & + \phi_1(p,x_3,z)\int_{x_3}^\infty \phi_2(p,y,z)g_2(p,y)dy] \;. \end{aligned}$$

Letting $\quad v_1(p,x_3,\sigma^2) = \lim\limits_{\substack{z\to\sigma^2 \\ \mathrm{Im}z>0}} v_1(p,x_3,z)$, we get

$$(4.23) \qquad \begin{aligned} v_1(p,x_3,\sigma^2) = i\chi_1(|p|)(c_S^2\xi_S(\sigma^2))^{-1}[&e^{i\xi_S(\sigma^2)x_3}\int_0^{x_3} \cos\xi_S(\sigma^2)y g_2(p,y)dy \\ &+ \cos\xi_S(\sigma^2)x_3 \int_{x_3}^{\infty} e^{i\xi_S(\sigma^2)y} g_2(p,y)dy]. \end{aligned}$$

From (4.19) we deduce that

$$(4.24) \qquad \begin{aligned} v_1(p,x_3,\sigma^2) = i\chi_1(|p|)(c_S^2\xi_S(\sigma^2))^{-1} \int_{x_3}^{\infty} [&\cos\xi_S(\sigma^2)x_3\, e^{i\xi_S(\sigma^2)y} \\ &-\cos\xi_S(\sigma^2)y\, e^{i\xi_S(\sigma^2)x_3}]g_2(p,y)dy. \end{aligned}$$

We have

$$(4.25) \qquad |e^{iax_3} - e^{iay}| \le 2^\gamma\, |a|^\gamma\, |x_3-y|^\gamma \quad \text{for} \quad 0 \le \gamma \le 1 .$$

Thus (4.24) and (4.25) imply

$$(4.26) \qquad |v_1(p,x_3,\sigma^2)| \le C\chi_1(|p|)(\xi_S(\sigma^2))^{\gamma-1} \int_{x_3}^{\infty} |x_3-y|^\gamma|g_2(p,y)|dy .$$

For α such that $0 \le \alpha < s_2-\gamma-1/2$ we have

$$(4.27) \qquad \int_{x_3}^{\infty} (1+y^2)^{\gamma-s_2}\, dy \le C(1+x_3^2)^{-\alpha} .$$

Therefore

$$(4.28) \qquad |v_1(p,x_3,\sigma^2)|^2 \le C\chi_1(|p|)(\xi_S(\sigma^2))^{2\gamma-2}(1+x_3^2)^{-\alpha} \int_0^{\infty} (1+y^2)^{s_2}|g_2(p,y)|^2dy .$$

Let $(|p|,\theta)$ be the polar coordinates of p. The first remark to be made is that $\int_{|p|=r} |g_2(p,y)|^2|p|d\theta$ has a meaning because $g_2(\cdot,y)$ is in $H^{s_1}(\mathbb{R}^2)$ for almost every $y > 0$.

The second remark to be made is that the function $|p| \to (\xi_S(\sigma^2))^{-\beta}$ is integrable on $[0,\sigma/c_S)$ for $\beta < 2$.

From (4.26) and these two remarks we deduce that $v_1(.,.,\sigma^2)$ verifies (4.18).

Remark

Just note that in the following the Young's inequality, the Peetre's inequality and this estimate :

$$(4.29)\qquad \int_0^\infty e^{-ay}(1+y)^s dy \le \Gamma(1+s)e^a\, a^{-(1+s)} \quad , \quad a > 0 \ , \ 1+s > 0$$

will be often used without mentioning them explicitely.

4.1.2. Definition and estimation of $v_2(p,x_3,\sigma^2)$

Let

$$\phi_1(p,x_3,z) = \mathrm{ch}\ \xi_S'(z)x_3 \quad \text{and} \quad \phi_2(p,x_3,z) = \exp(-\xi_S'(z)x_3)$$

We have

$$(4.30)\qquad \begin{aligned} v_2(p,x_3,z) &= \chi_2(|p|)(c_S^2\xi_S'(z))^{-1}[\phi_2(p,x_3,z)\int_0^{x_3}\phi_1(p,y,z)g_2(p,y)dy \\ &\quad + \phi_1(p,x_3,z)\int_{x_3}^\infty \phi_2(p,y,z)g_2(p,y)dy] \end{aligned}$$

and as in 4.1.1 we get

$$(4.31)\qquad \begin{aligned} v_2(p,x_3,\sigma^2) &= \chi_2(|p|)(c_S^2\xi_S'(\sigma^2))^{-1}[e^{-\xi_S'(\sigma^2)x_3}\int_0^{x_3}\mathrm{ch}\xi_S'(\sigma^2)y\ g_2(p,y)dy \\ &\quad + \mathrm{ch}\ \xi_S'(\sigma^2)x_3\int_{x_3}^\infty e^{-\xi_S'(\sigma^2)y}\ g_2(p,y)dy]. \end{aligned}$$

But in order to estimate (4.31) we cannot use (4.19) as for $v_1(.,.,\sigma^2)$ because we have $|p| > \sigma/c_S$ in the case of $v_2(.,.,\sigma^2)$. Instead we adapt to our case the method followed in the proof of theorem 3.5 of [6].

Let

$$p' = p\,|p|^{-1}\,(2\sigma^2 c_S^2 - |p|^2)^{1/2} = p\,|p|^{-1}\,(\sigma^2 c_S^{-2} - (\xi_S'(\sigma^2))^2)^{1/2} \tag{4.32}$$

and in order to simplify let

$$\xi_S(\sigma^2) = (\sigma^2 c_S^{-2} - |p'|^2)^{1/2} \quad \text{and} \quad \xi_S'(\sigma^2) = (|p|^2 - \sigma^2 c_S^{-2})^{1/2} .$$

One verifies that $|p'| < \sigma/c_S$ and $\xi_S(\sigma^2) = \xi_S'(\sigma^2)$.

From (4.19) and (4.31) we obtain

$$v_2(p,x_3,\sigma^2) = \sum_{j=1}^{4} v_{2j}(p,x_3,\sigma^2) \tag{4.33}$$

with

$$v_{2j}(p,x_3,\sigma^2) = \chi_2(|p|)(c_S^2\xi_S'(\sigma^2))^{-1}\, v_{2j}'(p,x_3,\sigma^2)$$

and

$$\begin{aligned}
v_{21}'(p,x_3,\sigma^2) &= e^{-\xi_S'(\sigma^2)x_3} \int_0^{x_3} [\mathrm{ch}\,\xi_S'(\sigma^2)y - \cos\xi_S(\sigma^2)y]\, g_2(p,y)\,dy ,\\
v_{22}'(p,x_3,\sigma^2) &= e^{-\xi_S'(\sigma^2)x_3} \int_0^{x_3} \cos\xi_S(\sigma^2)y\,[g_2(p,y) - g_2(p',y)]\,dy ,\\
v_{23}'(p,x_3,\sigma^2) &= \mathrm{ch}\,\xi_S'(\sigma^2)x_3 \int_{x_3}^{\infty} e^{-\xi_S'(\sigma^2)y}\,[g_2(p,y) - g_2(p',y)]\,dy ,\\
v_{24}'(p,x_3,\sigma^2) &= \int_{x_3}^{\infty} [\mathrm{ch}\,\xi_S'(\sigma^2)x_3\, e^{-\xi_S'(\sigma^2)y} - e^{-\xi_S'(\sigma^2)x_3} \cos\xi_S(\sigma^2)y]\\
&\qquad g_2(p',y)\,dy .
\end{aligned} \tag{4.34}$$

But we have the following inequalities

$$(4.35)\qquad |\mathrm{ch}\xi_S'(\sigma^2)y - \cos\xi_S(\sigma^2)y| \le M_\gamma(\xi_S'(\sigma^2))^{2\gamma}(1+y^2)^\gamma e^{\xi_S'(\sigma^2)y}, \quad 0 \le \gamma \le 1$$

$$(4.36)\qquad \int_{|p|=r} |g_2(p,y)-g_2(p',y)|^2 d\theta \le M_{\gamma,r}(\xi_S'(\sigma^2))^{4\gamma}\int(1+x_1^2+x_2^2)^{s_1}|f(x_1,x_2,y|^2 dx_1 dx_2$$

if $0 \le \gamma \le 1$ and $\gamma < s_1 - 1/2$,

$$(4.37)\qquad |\mathrm{ch}\xi_S'(\sigma^2)x_3\, e^{-\xi_S'(\sigma^2)y} - e^{-\xi_S'(\sigma^2)x_3}\cos\xi_S(\sigma^2)y| \le M_\gamma(\xi_S'(\sigma^2))^\gamma(1+y^2)^{\gamma/2}$$

if $0 \le \gamma \le 1$ and $0 \le x_3 \le y$.

From (4.35) - (4.37) we easily deduce that each v_{2j} verifies (4.18) with, respectively,

$\delta < s_2 - 1/2$ if $1/2 < s_2 \le 1$ for v_{21}

$\delta < \inf(s_2/2, 3/2)$ if $1 < s_2$ for v_{21}

$\delta < \inf(2s_1 - 1, 2)$ for v_{22}

$\delta < s_2 - 1/2$ for v_{23} and v_{24}.

Thus $v_2(.,.,\sigma^2)$ verifies (4.18).

4.1.3. Definition and estimation of $v_3(p,x_3,\sigma^2)$

$v_3(p,x_3,\sigma^2)$ is also given by (4.31) with $\chi_3(|p|)$ instead of $\chi_2(|p|)$. Estimation of $v_3(p,x_3,\sigma^2)$ is very simple because we have $\xi_S'(\sigma^2) \ge \sigma/c_S$. In fact $v_3(.,.,\sigma^2)$ verifies (4.18) with $\delta = s_2 + 1/2$.

As for the remarks 2.7 and 2.8 in [6] we have $R_2^+(\sigma^2)g_2 = R_2^-(\sigma^2)g_2$.

This concludes the proof of proposition 4.2. Q.E.D.

4.2. Study of $R_1(\sigma^2)g'$

Let η be a strictly positive number such that $\sigma/c_S+\eta < \sigma/c_R-\eta$. Let χ_1 (resp. χ_2, χ_3, χ_4, χ_5, χ) be the characteristic function of $(0,\sigma/c_P)$ (resp. $(\sigma/c_P, \sigma/c_S)$, $(\sigma/c_S,\sigma/c_S+\eta)$, $(\sigma/c_S+\eta,\sigma/c_R-\eta) \cup (\sigma/c_R+\eta,\infty)$, $(\sigma/c_R-\eta,\sigma/c_R+\eta)$, $(0,\infty) \setminus [\sigma/c_R-\eta,\sigma/c_R+\eta]$.

Let

(4.38) $$w_i(p,x_3,z) = \chi_i(|p|)R_1(z)\ {}^t(g_1(p,.),g_3(p,.))(x_3)$$

where $z \in \mathbb{C}\setminus[0,\infty)$ and $i = 1,2,3,4,5$, and

$$w(p,x_3,\sigma^2) = \chi(|p|)\ R_1^+(\sigma^2)\ {}^t(g_1(p,.),g_3(p,.))(x_3).$$

We then have

Proposition 4.3

Assume $s_1 > 1/2$ *and* $s_2 > 1/2$. *Let* $f \in L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ *and let* σ *be a strictly positive number such that*

(4.59) $$\tau_P(\sigma)\bar{f} = \tau_{SV}(\sigma)\bar{f} = \tau^0_{SV}(\sigma)f = 0\ .$$

Let δ *be a real number such that*

(4.40) $$\delta < s_2-1/2 \quad \text{if} \quad 1/2 < s_2 \leq 1$$

(4.41) $$\delta < s_2/2 \quad \text{if} \quad 1 < s_2 \leq 3/2$$

(4.42) $$\delta < \inf(s_2/2,2s_1,5/2) \quad \text{if} \quad 3/2 < s_2\ .$$

Then

$$w(.,.,\sigma^2) = \chi(|.|)R_1^+(\sigma^2)\ {}^t(g_1,g_3)$$

$$= \chi(|.|)R_1^-(\sigma^2)\ {}^t(g_1,g_3) \in L^2(\mathbb{R}^3_+,\mathbb{C}^3,(1+x_3^2)^{\delta-1/2}dpdx_3)$$

and

(4.43) $$||w(.,.,\sigma^2)||_{L^2(\mathbb{R}^3_+,\mathbb{C}^2,(1+x_3^2)^{\delta-1/2}dpdx_3} \leq M_\delta(\sigma)\ ||f||_{0;s_1,s_2}$$

where $M_\delta(.)$ *is a positive continuous function on* $(0,\infty)$.

Let $\mathbb{R}^2_p$ denote $\mathbb{R}^2$ when $p = (p_1,p_2)$ are the variables. We then have

Proposition 4.4

Suppose $s_1 > 1/2$ *and* $s_2 > 1/2$. *Let* $f \in L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ *and let* σ *be a strictly positive number such that*

(4.44) $$\tau_R(\sigma)f = 0\ .$$

Let δ *be a real number such that* $\delta < s_2$.
Then

$$w_5(.,.,\sigma^2) \in H^{\inf(s_1-1,0)}(\mathbb{R}^2_p,L^{2;\delta-1/2}(\mathbb{R}_+,\mathbb{C}^2,dx_3))$$

(4.45) $$||w_5(.,.,\sigma^2)||_{H^{\inf(s_1-1,0)}(\mathbb{R}^2_p,L^{2;\delta-1/2}(\mathbb{R}_+,\mathbb{C}^2,dx_3))} \leq M_\delta(\sigma)||f||_{0;s_1,s_2}$$

where $M_\delta(.)$ *is a positive continuous function on* $(0,\infty)$.

As a consequence of (4.18), (4.43) and (4.45) we have

(4.46) $$\chi_5(|p|)\Pi(R_1^+(\sigma^2)g'(p,.)\oplus R_2^+(\sigma^2)g_2(p,.))(x_3) \in H^{\inf(s_1-1,0)}(\mathbb{R}^2_p,L^{2;\delta-1/2}(\mathbb{R}_+,\mathbb{C}^2,dx_3))$$

and (4.7) will enable us to show that

(4.47) $$\mathscr{F}^{-1}[U(p_1,p_2)\chi_5(|p|)\Pi(R_1^+(\sigma^2)g'(p,.)\oplus R_2^+(\sigma^2)g_2(p,.))](x_1,x_2,x_3) \in L^{2;\tilde{s}_1,\delta-1/2}(\mathbb{R}^3_+,\mathbb{C}^3)$$

where $\tilde{s}_1$ is defined in theorem 4.1.

Thus theorem 4.1 will be a consequence of propositions 4.2-4.4.

Let $b_1(|p|)$ be the 2×2 system of ordinary differential operators given by the r.h.s. of (3.12). The $w_i(.,.,\sigma^2)$ depend on solutions of $b_1(|p|)\phi = \sigma^2\phi$.

More precisely let

$$(4.48) \qquad \phi_P(x_3,|p|,\sigma^2) = 2\pi\, e^{-i|p|x_1}(\sigma\rho_0\xi_P(\sigma^2))^{1/2}\psi_P((x_1,x_3),|p|,\sigma)$$

where $\psi_P((x_1,x_3),|p|,\sigma)$ is given by (3.70)

$$(4.49) \qquad \phi_S(x_3,|p|,\sigma^2) = 2\pi\, e^{-i|p|x_1}(\sigma\rho_0\xi_S(\sigma^2))^{1/2}\psi_S((x_1,x_3),|p|,\sigma)$$

where $\psi_S((x_1,x_3),|p|,\sigma)$ is given by (3.71)

$$(4.50) \qquad \phi_S^0(x_3,|p|,\sigma^2) = 2\pi\, e^{-i|p|x_1}(\sigma\rho_0\xi_S(\sigma^2))^{1/2}\psi_S^0((x_1,x_2),|p|,\sigma)$$

where $\psi_S^0((x_1,x_3),|p|,\sigma)$ is given by (3.72)

$$(4.51) \qquad \phi_R(x_3,|p|) = 2\pi\, e^{-i|p|x_1}\,\psi_R((x_1,x_3),|p|)$$

where $\psi_R((x_1,x_3),|p|)$ is given by (3.73).

Remark that the r.h.s. of (4.48)-(4.51) depend only on σ^2.

We have

$$(4.52) \qquad b_1(|p|)\,\phi_P(x_3,|p|,\sigma^2) = \sigma^2\phi_P(x_3,|p|,\sigma^2), \quad \text{etc...}$$

$$(4.53) \qquad b_1(|p|)\,\phi_R(x_3,|p|) = c_R^2\,|p|^2\,\phi_R(x_3,|p|)\ .$$

The following lemma is proved by computation

Lemma 4.5

Suppose $s_1 > 1/2$ *and* $s_2 > 1/2$. *Let* $f \in L^{2;s_1,s_2}(\mathbb{R}^3_+,\mathbb{C}^3)$ *and let* σ *be a strictly positive number. Let* $g(.)$ *be given by (4.8). Then*

(i) $\tau_P(\sigma)\bar{f} = 0$ *if and only if*

(4.54) $$\int_0^\infty [\phi_{P1}(x_3,|p|,\sigma^2)g_1(p,x_3)-\phi_{P2}(x_3,|p|,\sigma^2)g_3(p,x_3)]\,dx_3 = 0 \text{ for } |p|<\sigma/c_P$$

(ii) $\tau_{SV}(\sigma)\bar{f} = 0$ *if and only if*

(4.55) $$\int_0^\infty [\phi_{S1}(x_3,|p|,\sigma^2)g_1(p,x_3)-\phi_{S2}(x_3,|p|,\sigma^2)g_3(p,x_3)]\,dx_3 = 0 \text{ for } |p|<\sigma/c_P$$

(iii) $\tau^0_{SV}(\sigma)\bar{f} = 0$ *if and only if*

(4.56) $$\int_0^\infty [\phi_{S1}(x_3,|p|,\sigma^2)g_1(p,x_3)-\phi^0_{S2}(x_3,|p|,\sigma^2)g_3(p,x_3)]\,dx = 0 \text{ for } \sigma/c_P<|p|<\sigma/c_S$$

(iv) $\tau_R(\sigma)f = 0$ *if and only if*

(4.57) $$\int_0^\infty [\overline{\phi_{R1}(x_3,|p|)}g_1(p,x_3)+\overline{\phi_{R2}(x_3,|p|)}g_3(p,x_3)]\,dx_3 = 0 \text{ for } |p| = \sigma/c_R .$$

4.2.1. Definition and estimate of $w_1(p,x_3,\sigma^2)$

In this section let $\phi_P(x_3,|p|,z)$ (resp. $\phi_S(x_3,|p|,z)$) be the function we get from (4.48) (resp. (4.49)) by substituting $\xi_P(z)$ (resp. $\xi_S(z)$) for $\xi_P(\sigma)$ (resp. $\xi_S(\sigma)$).

Let

(4.58) $$\begin{aligned}\Phi_P(x_3,|p|,z) &= e^{i\xi_P(z)x_3}\, t(|p|,\xi_P(z))\\ \Phi_S(x_3,|p|,z) &= e^{i\xi_S(z)x_3}\, t(\xi_S(z),-|p|) .\end{aligned}$$

All the ϕ and Φ satisfy $b_1(|p|)\phi = z\phi$. By adapting the proof of the corollary XIII 3.13 in [9] to our case we obtain

$$w_1(p,x_3,z) = \sum_{j=1}^{4} w_{1j}(p,x_3,z) = \sum_{j=1}^{4} i\chi_1(|p|)\,\frac{1}{2z}\,w'_{1j}(p,x_3,z)$$

for $\operatorname{Re} z > c_P^2|p|^2$ and $\operatorname{Im} z > 0$ where

$$w'_{11}(p,x_3,z) = (\xi_P(z))^{-1}\Phi_P(x_3,|p|,z)\int_0^{x_3}[\phi_{P1}(y,|p|,z)g_1(p,y) - \phi_{P2}(y,|p|,z)g_3(p,y)]dy \tag{4.59}$$

$$w'_{12}(p,x_3,z) = (\xi_S(z))^{-1}\Phi_S(x_3,|p|,z)\int_0^{x_3}[\phi_{S1}(y,|p|,z)g_1(p,y) - \phi_{S2}(y,|p|,z)g_3(p,y)]dy \tag{4.60}$$

$$w'_{13}(p,x_3,z) = (\xi_P(z))^{-1}\phi_P(x_3,|p|,z)\int_{x_3}^{\infty}[\Phi_{P1}(y,|p|,z)g_1(p,y) - \Phi_{P2}(y,|p|,z)g_3(p,y)]dy \tag{4.61}$$

$$w'_{14}(p,x_3,z) = (\xi_S(z))^{-1}\phi_S(x_3,|p|,z)\int_{x_3}^{\infty}[\Phi_{S1}(y,|p|,z)g_1(p,y) - \Phi_{S2}(y,|p|,z)g_3(p,y)]dy\,. \tag{4.62}$$

If z tends to σ^2 with a positive imaginary part we get

$$w_1(p,x_3,\sigma^2) = \sum_{j=1}^{4} w_{1j}(p,x_3,\sigma^2) = \sum_{j=1}^{4} i\chi_1(|p|)\,\frac{1}{2\sigma^2}\,w'_{1j}(p,x_3,\sigma^2)$$

where the $w'_{1j}(p,x_3,\sigma^2)$ are given by the r.h.s. of (4.59)-(4.62) but with $\xi_P(\sigma^2),\phi_P(.,.,\sigma^2)$ ... instead of $\xi_P(z), \phi_P(.,.,z)$ ...

Moreover it follows from (4.54) and (4.55) that we have

$$w'_{11}(p,x_3,\sigma^2) = -(\xi_P(\sigma^2))^{-1}\Phi_P(x_3,|p|,\sigma^2)\int_{x_3}^{\infty}[\phi_{P1}(y,|p|,\sigma^2)g_1(p,y) - \phi_{P2}(y,|p|,\sigma^2)g_3(p,y)]\,dy \tag{4.63}$$

$$w'_{12}(p,x_3,\sigma^2) = -(\xi_S(\sigma^2))^{-1}\Phi_S(x_3,|p|,\sigma^2) \int_{x_3}^{\infty} [\phi_{S1}(y,|p|,\sigma^2)g_1(p,y) \tag{4.64}$$
$$- \phi_{S2}(y,|p|,\sigma^2)g_3(p,y)]dy .$$

Let us remark that

$$(4.65) \qquad |\phi_{Pi}(y,|p|,\sigma^2)| \le C_\gamma(\xi_P(\sigma^2))^\gamma(1+y^2)^{\sigma/2} \quad \text{for} \quad 0 \le \gamma \le 1, \; i=1,2 .$$

Thus from $w'_{13}(p,x_3,\sigma^2)$, $w'_{14}(p,x_3,\sigma^2)$, (4.63), (4.64) and (4.65) we get

$$(4.66) \qquad |w_{1j}(p,x_3,\sigma^2)| \le C\chi_1(|p|)(\xi_P(\sigma^2))^{\gamma-1}(1+y^2)^{-\tau/2}(\int_0^\infty (1+y^2)^{s_2}|g(p,y)|^2dy)^{1/2}$$

if $0 \le \gamma \le 1$ and $\tau < s_2-\gamma-1/2$.

Finally $w_1(p,x_3,\sigma^2)$ satisfies (4.43) with $\delta<s_2-1/2$.

4.2.2. Definition and estimate of $w_2(p,x_3,\sigma^2)$

We now consider analytic extensions of the functions $\phi(x_3,|p|,z)$ and $\Phi(x_3,|p|,z)$ from the upper halfplane $\{\mathrm{Im} z > 0\}$ into the lower one $\{\mathrm{Im} z < 0\}$ through the interval $(c_S^2|p|^2, c_P^2|p|^2)$. For that it suffices to substitute $i\xi'_P(z)$ for $\xi_P(z)$ in $\phi_P(x_3,|p|,z)$, $\phi_S(x_3,|p|,z)$, $\Phi_P(x_3,|p|,z)$ given in 4.2.1. $\phi_S(x_3,|p|,z)$ remains the same.

We keep the same notations for these analytic extensions as those used in 4.2.1 but we must remark that with these notations we now have

$$(4.67) \qquad \phi_S(x_3,|p|,\sigma^2) = \phi_S^0(x_3,|p|,\sigma^2)$$

where $\sigma/c_P < |p| < \sigma/c_S$ and where $\phi_S^0(x_3,|p|,\sigma^2)$ is given by (4.50).

Thus for $z \in \mathbb{C}$ such that $c_S^2|p|^2 < \mathrm{Re} z < c_P^2|p|^2$ and $\mathrm{Im} z > 0$ we have

$$(4.68)\qquad w_2(p,x_3,z) = \sum_{j=1}^{4} w_{2j}(p,x_3,z)\ .$$

Each w_{2j} is deduced from the corresponding w_{1j} defined in 4.2.1 by substituting in it $\chi_2(|p|)(2z\xi_P'(z))^{-1}$ (resp. $i\chi_2(|p|)\ (2z\xi_S(z))^{-1}$) for $i\chi_1(|p|)\ (2z\xi_P(z))^{-1}$ (resp. $i\chi_1(|p|)\ (2z\xi_S(z))^{-1}$).

As in 4.2.1 we define $w_2(p,x_3,\sigma^2)$ and $w_{2j}(p,x_3,\sigma^2)$ (j=1,2,3,4). (4.67) enables us to use (4.56) in order to transform w_{22} as we have done for w_{11} and w_{12} in 4.2.1. We have

$$(4.69)\qquad |\phi_{Pi}(x_3,|p|,\sigma)| \le C_\gamma(\xi_P'(\sigma^2))^\gamma(1+x_3^2)^{\gamma/2}\ e^{\xi_P'(\gamma^2)x_3}$$

$$(4.70)\qquad |\phi_{Si}(x_3,|p|,\sigma)| \le C_\gamma(\xi_S(\sigma^2))^\gamma(1+x_3^2)^{\gamma/2}$$

for $i = 1,2$ and $0 \le \gamma \le 1$.

(4.69) and (4.70) enable us to estimate each w_{2j} (j=2,3,4) as w_{1j} in 4.2.1.

w_{21} needs a more careful study, especially when $s_2 > 1$.

First of all for s_2,δ and γ such that $1/2 < s_2 \le 1$, $\delta < s_2-1/2$ and $0 \le \gamma \le 1$ one has

$$|w_{21}(p,x_3,\sigma^2)|\,(1+y^2)^{(\delta-1/2)/2} \le$$

$$(4.71)\qquad \frac{C\chi_2(|p|)}{(\xi_P'(\sigma^2))^{1-\gamma}} \int_0^{x_3} \frac{e^{-\xi_P'(\sigma^2)(x_3-y)}}{(1+|x_3-y|^2)^{(1/2-\gamma)/2}}\,(1+y^2)^{\gamma/2}|g(p,y)|\,dy$$

and in that case w_{21} has the same behaviour as w_{2j} (j=2,3,4).

When $s_2 > 1$ we proceed in the same way as we have done for $v_2(x_3,|p|,\sigma)$ in 4.1.2.

Let ε be a strictly positive sufficiently small number such that $\sigma/c_P+\varepsilon < \sigma/c_S$. Suppose $\sigma/c_P < |p| < \sigma/c_P+\varepsilon$ and let

$$(4.72) \qquad p' = p|p|^{-1}(\sigma^2 c_P^{-2}-(\xi_P(\sigma^2))^2)^{1/2} = p|p|^{-1}(2\sigma^2 c_P^{-2}-|p|^2)^{1/2} .$$

We have $|p'| < \sigma/c_P$.

Let

$$(4.73) \qquad \xi_P(\sigma^2,|p'|) = (\sigma^2/c_P^2-|p'|^2)^{1/2} , \quad \xi_P'(\sigma^2,|p|) = (|p|^2-\sigma^2/c_P^2)^{1/2}$$

$$(4.74) \qquad \xi_S(\sigma^2,|p|) = (\sigma^2/c_S^2-|p|^2)^{1/2} , \quad \xi_S(\sigma 2,|p'|) = (\sigma^2/c_S^2-|p'|^2)^{1/2} .$$

Let

$$(4.75) \qquad \tilde{\phi}(x_3,|p|,|p'|,\sigma^2) = i\ d(|p|,|p'|,\sigma^2)\,\phi_P(x_3,|p'|,\sigma^2)$$

where

$$d(|p|,|p'|,\sigma^2) = \frac{(\sigma^2/c_S^2-2|p'|^2)^2+4|p'|^2\xi_P(\sigma^2,|p'|)\xi_S(\sigma^2,|p'|)}{(\sigma^2/c_S^2-2|p|^2)^2+4i|p|^2\xi_P'(\sigma^2,|p|)\xi_S(\sigma^2,|p|)} .$$

It follows from (4.54) that w_{21} can be written as

$$(4.76) \qquad \begin{aligned} w_{21}(p,x_3,\sigma^2) &= \sum_{j=1}^{5} w_{21,j}(p,x_3,\sigma^2) \\ &= \sum_{j=1}^{5} \chi_2(|p|)(2\sigma^2\xi_P'(\sigma^2,|p|))^{-1}\ w_{21,j}'(p,x_3,\sigma^2) \end{aligned}$$

where

$$(4.77) \qquad \begin{aligned} w_{21,1}'(p,x_3,\sigma^2) &= \phi_P(x_3,|p|,\sigma^2)\int_0^{x_3}[\phi_{P1}(y,|p|,\sigma^2) \\ &\quad - \tilde{\phi}_1(y,|p|,|p'|,\sigma^2)]\ g_1(p,y)dy \end{aligned}$$

$$w'_{21,2}(p,x_3,\sigma^2) = \Phi_p(x_3,|p|,\sigma^2)\int_0^{x_3} [\tilde{\phi}_2(y,|p|,|p'|,\sigma^2) - \phi_{p2}(y,|p|,\sigma^2)]\ g_3(p',y)dy \tag{4.78}$$

$$w'_{21,3}(p,x_3,\sigma^2) = \Phi_p(x_3,|p|,\sigma^2) \int_0^{x_3} \tilde{\phi}_1(y,|p|,|p'|,\sigma^2) (g_1(p,y)-g_1(p',y))dy \tag{4.79}$$

$$w'_{21,4}(p,x_3,\sigma^2) = \Phi_p(x_3,|p|,\sigma^2) \int_0^{x_3} \tilde{\phi}_2(y,|p|,|p'|,\sigma^2) (g_3(p',y)-g_3(p,y))dy \tag{4.80}$$

$$w'_{21,5}(p,x_3,\sigma^2) = \Phi_p(x_3,|p|,\sigma^2) \int_{x_3}^{\infty} [\tilde{\phi}_2(y,|p|,|p'|,\sigma^2)g_3(p',y) - \tilde{\phi}_1(y,|p|,|p'|,\sigma^2)g_1(p',y)]\,dy\ . \tag{4.81}$$

Notations used in (4.77)-(4.81) should be clear. For example the r.h.s. of (4.77) is a function of both $|p|$ and $|p'|$ but it can be considered as a function of p alone because of (4.72). In this last case it is denoted by $w'_{21,1}(p,x_3,\sigma^2)$.

Finally it follows from (4.76)-(4.81) that w_{21} satisfies (4.43) for any δ such that $\delta < \inf(s_2/2\ ,\ 2s_1\ ,\ 2s_1+s_2-3/2\ ,\ 5/2)$ because of the following inequalities

$$||p|-|p'|| \le C(\xi'_p(\sigma^2,|p|))^2 \tag{4.82}$$

$$|\phi_{pi}(y,|p|,\sigma^2)-\phi_i(y,|p|,|p'|,\sigma^2)| \le C_\gamma(\xi'_p(\sigma^2,|p|)^{3\gamma}(1+y^2)^{3\gamma/2} e^{\xi'_p(\sigma^2,|p|)y} \tag{4.83}$$

$$|\tilde{\phi}_i(y,|p|,|p'|,\sigma^2)| \le C_\gamma(\xi'_p(\sigma^2,|p|))^\gamma(1+y^2)^{\gamma/2} \tag{4.84}$$

for $0 \le \gamma \le 1$.

4.2.3. Definition and estimation of $w_3(p,x_3,\sigma^2)$

In section 4.2.3 and 4.2.4 let $\phi_P(x_3,|p|,z), \phi_S(x_3,|p|,z)$ $\Phi_P(x_3,|p|,z)$ and $\Phi_S(x_3,|p|,z)$ be the functions we obtain from (4.48), (4.50) and (4.58) by substituting $\tilde{\xi}_P(z)$ and $\tilde{\xi}_S(z)$ for $\xi_P(\sigma^2)$ and $\xi_S(\sigma^2)$.

As before we have

$$w_3(p,x_3,z) = \sum_{j=1}^{4} w_{3j}(p,x_3,z) \ . \tag{4.85}$$

Each w_{3j} is deduced from the corresponding w_{1j} by using the new definitions of ϕ_P, ϕ_S, Φ_P, Φ_S and by substituting $\tilde{\xi}_P(z)$ (resp. $\tilde{\xi}_S(z)$, $\chi_3(|p|)$) for $\xi_P(z)$ (resp. $\xi_S(z)$, $\chi_1(|p|)$). When z tends to σ^2 with a positive imaginary part, we define $w_3(p,x_3,\sigma^2)$ and $w_{3j}(p,x_3,\sigma^2)$.

We now have $(i = 1,2)$

$$\begin{aligned} &|\Phi_{Pi}(x_3,|p|,\sigma^2)| \le C\, e^{-\xi_P'(\sigma^2)x_3} \\ &|\phi_{Pi}(x_3,|p|,\sigma^2)| \le C\, e^{\xi_P'(\sigma^2)x_3} \\ &\xi_P'(\sigma^2) \ge C > 0 \end{aligned} \tag{4.86}$$

(4.86) enables us to show that $w_{31}(p,x_3,\sigma^2)$ and $w_{33}(p,x_3,\sigma^2)$ both satisfy (4.43) with $\delta = s_2+1/2$. Furthermore we have

$$\begin{aligned} &|\Phi_{Si}(x_3,|p|,\sigma^2)| \le C\, e^{-\xi_S'(\sigma^2)x_3} \\ &|\phi_{Si}(x_3,|p|,\sigma^2)| \le C_\gamma(\xi_S'(\sigma^2))^\gamma(1+x_3^2)^{\gamma/2}\, e^{\xi_S'(\sigma^2)x_3} \end{aligned} \tag{4.87}$$

with $0 \le \gamma \le 1$.

(4.87) enables us to show that $w_{34}(p,x_3,\sigma^2)$ satifies (4.43) for any δ such that $\delta < s_2-1/2$.

When $1/2 < s_2 \le 1$ these estimates show that $w_{32}(p,x_3,\sigma^2)$ satisfies (4.43) for any δ such that $\delta < s_2-1/2$. When $s_2 > 1$ the study of w_{32} is the same as for w_{21} in 4.2.2. In the same way we consider a strictly positive sufficiently small number ε and suppose $\sigma/c_S < |p| < \sigma/c_S+\varepsilon$. Let

$$p' = p|p|^{-1}(2\sigma^2/c_S^2-|p|^2)^{1/2} = p|p|^{-1}(\sigma^2/c_S^2-(\xi_S'(\sigma^2))^2)^{1/2} \tag{4.88}$$

$$\xi_S'(\sigma^2,|p|) = (|p|^2-\sigma^2/c_S^2)^{1/2} \quad , \quad \xi_S(\sigma^2,|p'|) = (\sigma^2/c_S^2-|p'|^2)^{1/2} \tag{4.89}$$

$$\xi_P'(\sigma^2,|p|) = (|p|^2-\sigma^2/c_P^2)^{1/2} \quad , \quad \xi_P'(\sigma^2,|p'|) = (|p'|^2-\sigma^2/c_P^2)^{1/2} \ . \tag{4.90}$$

Let

$$\tilde{\phi}^0(x_3,|p|,|p'|,\sigma^2) = id(|p|,|p'|,\sigma^2)\ \phi_S^0(x_3,|p'|,\sigma^2) \tag{4.91}$$

where

$$d(|p|,|p'|,\sigma^2) = \frac{(\sigma^2/c_S^2-2|p'|^2)^2+4i|p'|^2\xi_P'(\sigma^2,|p'|)\xi_S(\sigma^2,|p'|)}{(\sigma^2/c_S^2-2|p|^2)^2-4|p|^2\xi_P'(\sigma^2,|p|)\xi_S'(\sigma^2,|p|)} \ . \tag{4.92}$$

Remark that $\tilde{\phi}^0(x_3,|p|,|p'|,\sigma^2)$ satisfies (4.56). Furthermore we have

$$|\tilde{\phi}_i^0(x_3,|p|,|p'|,\sigma^2)-\phi_{Si}(x_3,|p|,\sigma^2) \le C_\gamma(\xi_S'(\sigma^2,|p|))^{3\gamma}.(1+x_3^2)^{3\gamma/2} e^{\xi_S'(\sigma^2,|p|)x_3} \tag{4.93}$$

when $0 \le \gamma \le 1$.

Finally following the same method as in 4.2.2 we show that $w_{32}(p,x_3,\sigma^2)$ satisfies (4.43) for any δ such that $\delta < \inf(s_2/2, 2s_1, 2s_1+s_2-3/2, 5/2)$ when $s_2 > 1$.

4.2.4. Definition and estimate of $w_4(p,x_3,\sigma^2)$

The study of $w_4(p,x_3,\sigma^2)$ is very easy because when $\chi_4(|p|)$ is different from zero, σ^2 belongs to the resolvent set of $B_1(|p|)$ and $\xi_P'(\sigma^2)$ and $\xi_S'(\sigma^2)$ are larger than a strictly positive constant. $w_4(p,x_3,z)$ can be written in the same way as $w_3(p,x_3,z)$ with χ_4 instead of χ_3. Finally we verify that $w_4(p,x_3,\sigma^2)$ satisfies (4.43) for $\delta = s_2+1/2$.

4.2.5. $\chi(|\cdot|)R_1^+(\sigma^2)\ {}^t(g_1,g_2) = \chi(|\cdot|)R_1^-(\sigma^2)\ {}^t(g_1,g_2)$

is proved as in [6] (see remarks 2.7 and 2.8).

This concludes the proof of proposition 4.3. Q.E.D.

4.2.6. Proof of proposition 4.4

w_5 is dealt with in the same way as $\tilde{u}_4(p,y,\mu)$ in section III.4 of [6] (see (3.138)). Recall (4.9) and (4.51).

Let

$$w_5(p,x_3,z) = \chi_5(|p|)\tilde{g}_R'(p)\phi_R(x_3,|p|)(c_R^2|p|^2-z)^{-1} + u(p,x_3,z) \tag{4.94}$$

where

$$u(p,x_3,z) = w_5(p,x_3,z) - \lim_{z'\to c_R^2|p|^2} (c_R^2|p|^2-z')(c_R^2|p|^2-z)^{-1}w_5(p,x_3,z')$$

$$\tilde{g}_R'(p) = \int_0^\infty g'(p,x_3)\cdot\overline{\phi_R(x_3,|p|)}dx_3 \tag{4.95}$$

When z is not equal to $c_R^2|p|^2$, z is in the resolvent set of $B_1(|p|)$

and $w_5(p,x_3,z)$ can be written in the same way as $w_3(p,x_3,z)$ (see 4.2.3) but with χ_5 instead of χ_3.

Let

$$(4.96)\qquad D(z) = (z/c_S^2-2|p|^2)^2-4|p|^2\xi_P'(z)\xi_S'(z) = (z-c_R^2|p|^2)D'(z) .$$

Note that $D'(c_R^2|p|^2) \neq 0$.

Let

$$(4.97)\qquad \begin{aligned} \Gamma_P(x_3,|p|,z) &= D(z)\phi_P(x_3,|p|,z) \\ \Gamma_S(x_3,|p|,z) &= D(z)\phi_S(x_3,|p|,z) \end{aligned}$$

where $\phi_P(x_3,|p|,z)$ and $\phi_S(x_3,|p|,z)$ are defined in 4.2.3. $u(p,x_3,z)$ can be written as follows

$$(4.98)\qquad u(p,x_3,z) = \sum_{j=1}^{4} u_j(p,x_3,z)$$

where

$$(4.99)\qquad u_j(p,x_3,z) = \chi_5(|p|)(2(z-c_R^2|p|^2))^{-1}(u_{j1}'(p,x_3,z)-u_{j2}'(p,x_3,z)) .$$

Each u_{ji}' is defined as follows. Let

$$(4.100)\qquad \begin{aligned} e_P(z) &= (z\xi_P'(z)D'(z))^{-1} \\ e_S(z) &= (z\xi_S'(z)D'(z))^{-1} . \end{aligned}$$

Then

$$u'_{11}(p,x_3,z) = \int_0^{x_3} [\, e_P(z)\Phi_P(x_3,|p|,z)\Gamma_{P1}(y,|p|,z) - e_P(c_R^2|p|^2) \tag{4.101}$$
$$\Phi_P(x_3,|p|,c_R^2|p|^2)\Gamma_{P1}(y,|p|,c_R^2|p|^2)]\, g_1(p,y)dy$$

u'_{12} is deduced from u'_{11} by substituting Γ_{P2} and g_3 for Γ_{P1} and g_1. u'_{21} (resp. u'_{22}) is deduced from u'_{11} (resp. u'_{12}) by substituting the subscript S for P.

Moreover

$$u'_{31}(p,x_3,z) = \int_{x_3}^{\infty} [\, e_P(z)\Gamma_P(x_3,|p|,z)\Phi_{P1}(y,|p|,z) - \tag{4.102}$$
$$- e_P(c_R^2|p|^2)\Gamma_P(x_3,|p|,c_R^2|p|^2)$$
$$\Phi_{P1}(y,|p|,c_R^2|p|^2)]\, g_1(p,y)dy$$

u'_{32} is deduced from u'_{31} by substituting in it Φ_{P2} and g_3 for Φ_{P1} and g_1.

u'_{41} (resp. u'_{42}) is deduced from u'_{31} (resp. u'_{32}) by substituting in it the subscript S for P.

$u'_{ji}(p,x_3,\sigma^2)$ is defined as the limit of $u'_{ji}(p,x_3,z)$ when z tends to σ^2 with a positive imaginary part. Let

$$u_j(p,x_3,\sigma^2) = \chi_5(|p|)(2(\sigma^2-c_R^2|p|^2))^{-1}(u'_{j1}(p,x_3,\sigma^2)-u'_{j2}(p,x_3,\sigma^2)) \tag{4.103}$$

$$u(p,x_3,\sigma^2) = \sum_{j=1}^{4} u_j(p,x_3,\sigma^2)\ . \tag{4.104}$$

As in theorem 3.6 of [6] one shows that $u(p,x_3,\sigma^2)$ satisfies (4.43) with $\delta < s_2$.

(4.57) shows that $\tilde{g}'_R(p) = 0$ for $|p| = \sigma/c_R$. From this we deduce that $w_5(.,.,\sigma^2)-u(.,.,\sigma^2)$ belongs to $H^{s_1-1}(\mathbb{R}^2_p, L^{2,s}(\mathbb{R}_+,\mathbb{C}^2,dx_3))$ for any s by using decreasing properties of $\phi_R(x_3,|p|)$ (see Lemma 3.7 of [6]).

This concludes the proof of proposition 4.4. Q.E.D.

5. LIMITING ABSORPTION PRINCIPLE FOR A

In this section we give two proofs of the limiting absorption principle for A in appropriate Hilbert spaces. The first one uses the method of D.M. Eidus ([12], [13]) and the theorem 4.1. The second one uses the method of R. Phillips [22] (see also [20]) and the theorem 4.1 too. Phillips method enables us to prove that the extended resolvents of A are locally Hölder continuous as functions of z with values in bounded operators on appropriate spaces (see theorem 5.2).

Let us mention the main differences with the case of perturbed stratified media [6]. First of all the boundary of Ω is unbounded for the perturbed isotropic half space with a free boundary. Furthermore if $\psi(.)$ is in $C_0^\infty(\mathbb{R}^3)$ and if $\psi(x) = 1$ for $|x| \le L$ and if u is locally in the domain of A, it is not necessary true that ψu is in the domain of A because ψu does not necessary satisfy the free boundary condition.

This difficulty is solved by the following lemma.

Lemma 5.1

There exists a bounded linear transformation $\mathscr{E}$ *of* $H^{3/2}(\mathbb{R}^2,\mathbb{C}^3)$ *into* $H^3(\mathbb{R}^3,\mathbb{C}^3)$ *such that for every* $h \in H^{3/2}(\mathbb{R}^2,\mathbb{C}^3)$ *the extension* $u = \mathscr{E}h$ *satisfies*

(5.1) $$\mu_0\varepsilon_{13}(u)|_{x_3=0} = h_1 \ , \ \mu_0\varepsilon_{23}(u)|_{x_3=0}=h_2 \ , \ \sigma_{33}(u)|_{x_3=0} = h_3 .$$

If ε *is a strictly positive number and if* supp h *is compact the extension* $u = \mathscr{E}h$ *can be choosen such that*

(5.2) $$\operatorname{supp} u \subset \{x \in \mathbb{R}^3; d(x',\operatorname{supp} h) < \varepsilon \ , \ |x_3| < 1\}$$

where $x' = (x_1,x_2)$ *and* d *is the euclidian distance on* $\mathbb{R}^2$.

<u>Proof of the lemma</u>

Let

(5.3) $$\gamma_0(v_i) = v_i|_{x_3=0} \quad , \quad \gamma_1(v_i) = \frac{\partial v_i}{\partial x_3}|_{x_3=0}$$

for $v \in H^3(\mathbb{R}^3,\mathbb{C}^3)$.

Any such extension v of h must satisfy the following system

(5.4) $$\begin{aligned} &\mu_0(\gamma_1(v_1) + \gamma_0(\frac{\partial v_3}{\partial x_1})) = h_1 \\ &\mu_0(\gamma_1(v_2) + \gamma_0(\frac{\partial v_3}{\partial x_2})) = h_2 \\ &\lambda_0\gamma_0(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}) + (\lambda_0+2\mu_0)\gamma_1(v_3) = h_3 . \end{aligned}$$

(5.4) has an infinite number of solutions. In order to choose one of them we add the following conditions

(5.5) $$\gamma_0(v_1) = \gamma_0(v_2) = \gamma_0(v_3) = 0 .$$

Let $\alpha \in C_0^\infty(\mathbb{R})$ such that $\alpha(0) = 0$ and $\alpha'(0) = 1$. The solution u of (5.4) and (5.5) is given by

(5.6) $$(\mathscr{F}u)_i(p,x_3) = \mu_0^{-1}(1+|p|^2)^{-1/2}(\mathscr{F}h)_i(p)\alpha((1+|p|^2)^{1/2}x_3), \quad i=1,2$$

$$(5.7) \qquad (\mathcal{F}u)_3(p,x_3) = (\lambda_0+2\mu_0)^{-1}(1+|p|^2)^{-1/2}(\mathcal{F}h)_3(p)\alpha((1+|p|^2)^{1/2}x_3)$$

and the transformation $\mathcal{E}$ defined by

$$(5.8) \qquad \mathcal{E}h = u$$

is a bounded operator from $H^{3/2}(\mathbb{R}^2,\mathbb{C}^3)$ into $H^3(\mathbb{R}^3,\mathbb{C}^3)$ such that (5.1) is satisfied.

Moreover if supp h is compact one can choose $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\phi \equiv 1$ in a sufficiently small neighbourhood of supp h, then $\mathcal{E}h = \phi u$ satisfies both (5.1) and (5.2). Q.E.D.

Let s_1 and s_2 be two real numbers. Let $L^{2;s_1,s_2}(\Omega,\mathbb{C}^3)$ be the space of all measurable $\mathbb{C}^3$-valued functions on Ω defined by

$$(5.8) \qquad L^{2;s_1,s_2}(\Omega,\mathbb{C}^3) = \{u(x) : (1+x_1^2+x_2^2)^{s_1/2}(1+x_3^2)^{s_2/2}u(x) \in L^2(\Omega,\mathbb{C}^3)\} .$$

In $L^{2;s_1,s_2}(\Omega,\mathbb{C}^3)$ we introduce the norm

$$(5.9) \qquad |||u|||^2_{0;s_1,s_2} = \int_\Omega (1+x_1^2+x_2^2)^{s_1}(1+x_3^2)^{s_2}\, u(x).\overline{u(x)}dx .$$

More generally we consider weighted Sobolev L^2 spaces $H^{m;s_1,s_2}(\Omega, \mathbb{C}^3)$ defined for any integer $m \geq 0$ by

$$(5.10) \qquad H^{m;s_1,s_2}(\Omega,\mathbb{C}^3) = \{u(x); D^\alpha u \in L^{2;s_1,s_2}(\Omega,\mathbb{C}^3) \quad , \quad 0 \leq |\alpha| \leq m\}$$

$H^{m;s_1,s_2}(\Omega,\mathbb{C}^3)$ is a Hilbert space under the norm

$$(5.11) \qquad |||u|||^2_{m;s_1,s_2} = \sum_{|\alpha|\leq m} |||D^\alpha u|||^2_{0;s_1,s_2} .$$

Let

$$(5.12)\qquad H^{1;s_1,s_2}(\Omega,\mathscr{A},\mathbb{C}^3) = \{u \in H^{1;s_1,s_2}(\Omega,\mathbb{C}^3);\, \mathscr{A}u \in L^{2;s_1,s_2}(\Omega,\mathbb{C}^3)\}$$

where $\mathscr{A}$ is defined by (2.11).

In $H^{1;s_1,s_2}(\Omega,\mathscr{A},\mathbb{C}^3)$ we introduce the norm

$$(5.13)\qquad |||u|||^2_{\mathscr{A};s_1,s_2} = |||u|||^2_{1;s_1,s_2} + |||\mathscr{A}u|||^2_{0;s_1,s_2}\ .$$

Suppose $s_1 \geq 0$ and $s_2 \geq 0$. Let us remark that the l.h.s. of (2.12) has still a meaning for u in $H^{1;-s_1,-s_2}(\Omega,\mathscr{A},\mathbb{C}^3)$ and v in $H^{1;s_1,s_2}(\Omega,\mathbb{C}^3)$. Thus u in $H^{1;-s_1,-s_2}(\Omega,\mathscr{A},\mathbb{C}^3)$ is said to satisfy the *generalized free boundary condition* if (2.12) is verified for every v in $H^{1;s_1,s_2}(\Omega,\mathbb{C}^3)$.

Finally let

$$(5.14)\qquad \begin{aligned} H^{1;-s_1,-s_2}(\Omega,A,\mathbb{C}^3) = \{&u \in H^{1;-s_1,-s_2}(\Omega,\mathscr{A},\mathbb{C}^3);\\ &u \text{ satisfies the generalized free boundary condition}\}\end{aligned}$$

$H^{1;-s_1,-s_2}(\Omega,A,\mathbb{C}^3)$ is a closed subspace of $H^{1;-s_1,-s_2}(\Omega,\mathscr{A},\mathbb{C}^3)$.

We then have

<u>Theorem 5.2</u> (Limiting absorption principle for A)

Suppose $s_1 > 1/2$ *and* $s_2 > 1/2$.

(i) For any $\omega_0 > 0$ *such that* ω_0^2 *is not an eigenvalue for* A *the following two limits exist in the uniform operator topology of* $B(L^{2;s_1,s_2}(\Omega,\mathbb{C}^3),H^{1;-s_1,-s_2}(\Omega,A,\mathbb{C}^3))$:

(5.15) $$R^{\pm}(\omega_0^2) = \lim_{\substack{z\to\omega_0^2 \\ \pm \mathrm{Im} z\ 0}} R(z) .$$

(ii) For any compact interval $[a,b]$ *in* $(0,\infty)$ *which does not contain any eigenvalue of* A *let*

(5.16) $$J^{\pm}(a,b) = \{z \in \mathbb{C}; \mathrm{Re} z \in [a,b] \text{ and } \pm \mathrm{Im} z \in [0,1]\}$$

(5.17) $$R^{\pm}(z) = R(z) \text{ if } \mathrm{Im} z \neq 0 \text{ and } z \in J^{\pm}(a,b).$$

For every δ *such that* $\delta < \inf(1, s_1 - 1/2, s_2 - 1/2)$ *there exists a positive constant* C *depending on* $A, a, b,$ *and* δ *such that*

(5.18) $$||R^+(z_1) - R^+(z_2)||_{B(L^{2;s_1,s_2}(\Omega,\mathbb{C}^3), H^{1;-s_1,-s_2}(\Omega,A,\mathbb{C}^3))} \leq C|z_1 - z_2|^{\delta}$$

(resp.

(5.19) $$||R^-(z_1) - R^-(z_2)||_{B(L^{2;s_1,s_2}(\Omega,\mathbb{C}^3), H^{1;-s_1,-s_2}(\Omega,A,\mathbb{C}^3))} \leq C|z_1 - z_2|^{\delta})$$

for any $(z_1, z_2) \in J^+(a,b) \times J^+(a,b)$ (resp. $J^-(a,b) \times J^-(a,b)$) .

We will give two differents proofs of (i). The first one uses the method of D.M. Eidus and the second one follows the R. Phillips method. (ii) is proved by using the Phillips one.

First proof

In order to prove (i) by using the method of D.M. Eidus one knows that the main result to be proved (see [30]) is the following inequality

$$(5.20)\qquad |||R(z)f|||_{\mathcal{A};-s_1,-s_2} \le C\ |||f|||_{0;s_1,s_2}$$

for every f in $L^{2;s_1,s_2}(\Omega,\mathbb{C}^3)$ and z in $J^{\pm}(a,b)\backslash[a,b]$.

Several steps are needed.

First step

Suppose (5.20) false. Then there exist sequences (u_n) in $D(A)$, (F_n) in $L^{2;s_1,s_2}(\Omega,\mathbb{C}^3)$ and (z_n) in $J^{\pm}(a,b)\backslash[a,b]$ such that

$$(5.21)\qquad z_n \text{ tends to } \sigma^2 \in [a,b]$$

$$(5.22)\qquad |||u_n|||_{\mathcal{A};-s_1,-s_2} = 1$$

$$(5.23)\qquad |||F_n|||_{0;s_1,s_2} < 1/n$$

$$(5.24)\qquad (A-z_n)u_n = F_n\ .$$

By choosing $s_1' > s_1$ and $s_2' > s_2$ and by considering, if it is necessary, a subsequence of (u_n) we denote by the same symbol, one then shows that (u_n) converges to a limit, denoted by u, in $H^{1;-s_1',-s_2'}(\Omega,A,\mathbb{C}^3)$. This is a consequence of Korn's inequality and Rellich theorem. Finally we obtain

$$(5.25)\qquad (\mathcal{A}-\sigma^2)u = 0\ .$$

Let

$$(5.26)\qquad \Omega' = \Omega \cap \{x \in \mathbb{R}^3;\ |x| > L\}\ .$$

Then u is in $H^{2;-s_1',-s_2'}(\Omega',\mathbb{C}^3)$ (see [21] p. 222) and the traces of $\varepsilon_{13}(u)$, $\varepsilon_{23}(u)$ and $\sigma_{33}(u)$ on $\Omega' \cap \{x \in \mathbb{R}^3; x_3=0\}$ are well defined and equal to zero.

Second step

Let ϕ be a function in $C^\infty(\mathbb{R}^3)$ such that $\phi(x) = 1$ for $|x| > L+2$ and $\phi(x) = 0$ for $|x| < L+1$. Let

$$(5.27) \qquad \mu_0\, \varepsilon_{13}(\phi u)|_{x_3=0} = h_1 \;,\quad \mu_0\varepsilon_{23}(\phi u)|_{x_3=0} = h_2 \;,\quad \sigma_{33}(\phi u)|_{x_3=0} = h_3 .$$

It follows from lemma 5.1 that there exists an extension $\tilde{u}$ of h such that the support of $\tilde{u}$ is compact and contained in $\{x \in \overline{\mathbb{R}^3_+}; |x| > L\}$.

Let

$$(5.28) \qquad u' = \phi u - \tilde{u}$$

u' in $H^{1;-s_1',-s_2'}(\Omega,\mathscr{A},\mathbb{C}^3)$ satisfies (2.12) for every v in $H^{1;s_1',s_2'}(\Omega,\mathbb{C}^3)$. Let $\mathscr{A}_0$ be the following operator in $\mathscr{D}'(\mathbb{R}^3_+,\mathbb{C}^3)$:

$$(5.29) \qquad \mathscr{A}_0 u = -\left(\frac{\lambda_0+\mu_0}{\rho_0}\right)\nabla(\nabla.u) - \frac{\mu_0}{\rho_0}\Delta u \;,\; u \in \mathscr{D}'(\mathbb{R}^3_+,\mathbb{C}^3) \quad .$$

Then it follows from (5.25) that $(\mathscr{A}_0-\sigma^2)u'$ is in $L^{2;s,t}(\mathbb{R}^3_+,\mathbb{C}^3)$ for any real numbers s and t.

From the sequence (z_n) there exists a subsequence we denote by the same symbol such that either $\mathrm{Im} z_n > 0$ or $\mathrm{Im} z_n < 0$.

Suppose $\mathrm{Im} z_n > 0$. It then follows from (5.23), (5.24) and Lemma 5.1 that one has

$$(5.30) \qquad u' = R_0^+(\sigma^2)(\mathscr{A}_0-\sigma^2)u' \; .$$

(If $\mathrm{Im} z_n < 0$ for every n we substitute $R_0^-(\sigma^2)$ for $R_0^+(\sigma^2)$ in (5.30)).

Third step

In fact one has (see (3.53))

$$\tau_P(\sigma)[(\mathcal{A}_0-\sigma^2)u'] = \tau_{SV}(\sigma)[(\mathcal{A}_0-\sigma^2)u'] = \tau^0_{SV}(\sigma)[(\mathcal{A}_0-\sigma^2)u'] = \tag{5.31}$$
$$= \tau_{SH}(\sigma)[(\mathcal{A}_0-\sigma^2)u'] = \tau_R(\sigma)[(\mathcal{A}_0-\sigma^2)u'] = 0 .$$

It then follows from (5.30), theorem 4.1 and (5.31) that u' is in $L^2(\mathbb{R}^3_+,\mathbb{C}^3)$. Therefore u is in $L^2(\Omega,\mathbb{C}^3,\rho(x)dx)$.

Duality between $L^{2;-s_1',s_2'}(\Omega,\mathbb{C}^3)$ and $L^{2;s_1',s_2'}(\Omega,\mathbb{C}^3)$ is denoted by $\langle\cdot,\cdot\rangle$.

From lemma 3.10 and (5.30) it is sufficient to show that

$$I = \langle\overline{R_0^+(\sigma^2)[(\mathcal{A}_0-\sigma^2)u']},(\mathcal{A}_0-\sigma^2)u'\rangle = \langle\bar{u}',(\mathcal{A}_0-\sigma^2)u'\rangle \tag{5.32}$$

is a real number in order to prove (5.31).

Remark that the support of $(\mathcal{A}_0-\sigma^2)u'$ is contained in $\{x \in \mathbb{R}^3_+; |x| < L+2\}$. Let χ be a real function in $C_0^\infty(\mathbb{R}^3)$ such that $\chi(x) = 1$ for $|x| < L+2$. We then have

$$I = \langle\chi\bar{u}',\mathcal{A}_0u'\rangle - \sigma^2\langle\bar{u}',\chi u'\rangle . \tag{5.33}$$

Thus, in order to show that I is a real number it is sufficient to prove that $\langle\chi\bar{u},\mathcal{A}_0u'\rangle$ is real.

u' satisfies the generalized free boundary condition. Thus we get

$$I = \int_{\mathbb{R}^3_+} [\lambda_0\nabla.u'(\bar{u}'.\nabla\chi) + \mu_0 \sum_{i,j=1}^3 \varepsilon_{ij}(u')(\bar{u}'_i\frac{\partial\chi}{\partial x_j} + \bar{u}'_j\frac{\partial\chi}{\partial x_i})]dx + G \tag{5.34}$$

where G is a real number.

In the r.h.s. of (5.34) we are integrating on the support of $\nabla\chi$.

Thus one can substitute u for u' in (5.34).

But we have $\mathscr{A}u = \sigma^2 u$ and u is in $H^{1;-s_1',-s_2'}(\Omega,A,\mathbb{C}^3)$. Therefore $<\chi\bar{u},\mathscr{A}u>$ is a real number and we have

$$J = <\chi\bar{u},\mathscr{A}u> = \int_\Omega [\lambda\nabla.u\ \nabla.(\chi\bar{u}) + 2\mu \sum_{i,j=1}^{3} \varepsilon_{ij}(u)\varepsilon_{ij}(\chi\bar{u})]dx$$

(5.35)

$$= \int_{\mathbb{R}^3_+} [\lambda\nabla.u(\bar{u}.\nabla\chi) + \mu \sum_{i,j=1}^{3} \varepsilon_{ij}(u)(\overline{u_i}\frac{\partial\chi}{\partial x_j} + \overline{u_j}\frac{\partial\chi}{\partial x_i})]dx + G'$$

where G' is a real number.

One has $\lambda(x) = \lambda_0$ and $\mu(x) = \mu_0$ for x in the support of $\nabla\chi$. Furthermore the r.h.s. of (5.35) is real. Therefore I is real too. Thus (5.31) is proved.

Finally we prove that u is different from zero because we have

(5.36) $$|||u|||_{\mathscr{A};-s_1,-s_2} = 1 .$$

Thus u is an eigenfunction of A and σ^2 in $[a,b]$ is the associated eigenvalue. This is a contradiction because we have supposed that the interval $[a,b]$ does not contain any eigenvalue of A.

In order to finish the proof of (i) see [6] or [30].

Second proof

Let $\overline{\mathbb{C}^+} = \{z \in \mathbb{C}; z \neq 0 , \operatorname{Im}z \geq 0\}$. In this new approach we give a construction of $u = R(z)f$ for every f in $L^{2;s_1,s_2}(\Omega)$ and every z in $\overline{\mathbb{C}^+}$ such that $\operatorname{Im}z > 0$.

From this construction it is possible to deduce (i) and (ii). Several steps are needed for proving (i) and (ii) of theorem 5.2.

First step

Let g in $L^{2;s_1,s_2}(\Omega)$. Define $\tilde{g}$ in $L^{2;s_1,s_2}(\mathbb{R}^3_+)$ as follows :

(5.37)
$$\tilde{g}(x) = g(x) \quad \text{if} \quad x \in \mathbb{R}^3_+ \cap \Omega$$
$$\tilde{g}(x) = 0 \quad \text{if} \quad x \in \mathbb{R}^3_+ \setminus (\mathbb{R}^3_+ \cap \Omega) .$$

Let

(3.58)
$$\Omega^L_+ = \{x \in \Omega; |x| < L+6 \ , \ x_3 > 0\}$$
$$\Omega^L_- = \{x \in \Omega; |x| < L+6 \ , \ x_3 \le 0\}$$
$$\Omega^L = \Omega^L_+ \cup \Omega^L_- .$$

Let

(5.39)
$$u_0 = R_0(z)\tilde{g} \quad \text{if} \quad \mathrm{Im} z > 0$$
$$u_0 = R^+_0(z)\tilde{g} \quad \text{if} \quad z \in (0,\infty) .$$

Let $\mathcal{A}_L$ be the restriction of $\mathcal{A}$ to Ω^L. v in $H^1(\Omega^L,\mathbb{C}^3)$ satisfies the generalized free boundary condition in Ω^L if and only if

(5.40)
$$\int_{\Omega^L} (\mathcal{A}_L v)_i \, \overline{w_i} \rho dx = \int_{\Omega^L} [\lambda(\nabla.v)(\nabla.\overline{w}) + 2\mu \ \varepsilon_{ij}(v)\varepsilon_{ij}(\overline{w})]dx$$

for every w in $H^1(\Omega^L,\mathbb{C}^3)$.

The following operator $(A_L, D(A_L))$:

$$D(A_L) = \{v \in H^1(\Omega^L,\mathbb{C}^3); \mathscr{A}_L v \in L^2(\Omega^L,\mathbb{C}^3) \text{ and } v \text{ satisfies (5.40)}\}$$

(5.41)

$$A_L u = \mathscr{A}_L u \quad , \quad u \in D(A_L)$$

is a positive selfadjoint one in $L^2(\Omega^L,\mathbb{C}^3,\rho dx)$.

Let ψ_1 and ψ_2 be two functions in $C_0^\infty(\mathbb{R}^3)$ such that

$$(5.42) \qquad \psi_1(x) = \begin{cases} 0 & \text{if } |x| > L+2 \\ 1 & \text{if } |x| < L+1 \end{cases}$$

$$(5.43) \qquad \psi_2(x) = \begin{cases} 0 & \text{if } |x| > L+5 \\ 1 & \text{if } |x| < L+4 \ . \end{cases}$$

The lemma 5.1 shows that there exists $\tilde{u}_0$ in $H^3(\mathbb{R}^3,\mathbb{C}^3)$ such that $\psi_2 u_0 - \tilde{u}_0$ satisfies the generalized free boundary condition on $\partial\mathbb{R}^3_+$. $\tilde{u}_0$ can be choosen in such a way that its support is contained in $\{x \in \mathbb{R}^3; L+3 < |x| < L+6\}$.

Let Λ be a complex number such that $\operatorname{Im}\Lambda < 0$. Let

$$(5.44) \qquad w = (\mathscr{A}_0 - \Lambda)(\psi_2 u_0 - \tilde{u}_0) \quad \text{in } \Omega^L_+$$

$$(5.45) \qquad w = g \quad \text{in } \Omega^L_- \ .$$

Let

$$(5.46) \qquad w' = (A_L - \Lambda)^{-1} w \ .$$

We then have

$$(5.47) \qquad (A_L - \Lambda) w' = (\mathscr{A}_0 - \Lambda)(\psi_2 u_0 - \tilde{u}_0) \quad \text{in } \Omega^L_+$$

$$(5.48) \qquad (A_L - \Lambda) w' = g \quad \text{in } \Omega^L_- \ .$$

Second step

Let

$$h_1 = \mu_0\ \varepsilon_{13}((1-\psi_1)u_0 + \psi_1 w')|_{x_3=0}$$

$$(5.49)\qquad h_2 = \mu_0\ \varepsilon_{23}((1-\psi_1)u_0 + \psi_1 w')|_{x_3=0}$$

$$h_3 = \sigma_{33}((1-\psi_1)u_0 + \psi_1 w')|_{x_3=0}\ .$$

The support of $h = {}^t(h_1,h_2,h_3)$ is contained in $\{x \in \mathbb{R}^3; L+1 < |x| < L+2\}$. One easily verifies that h is in $H^{3/2}(\mathbb{R}^2,\mathbb{C}^3)$. It follows from Lemma 5.1 that there exists $\tilde{u} = \mathscr{E}(h)$ such that

$$(5.50)\qquad ||\tilde{u}||_{H^3(\mathbb{R}^3,\mathbb{C}^3)} \le M\ |||g|||_{0;s_1,s_2}$$

for some positive constant M.

Furthermore $\tilde{u}$ can be choosen in such way that its support is contained in $\{x \in \mathbb{R}^3; L < |x| < L+3\}$.

Third step

For any z in $\overline{\mathbb{C}^+}$ let

$$(5.51)\qquad \begin{aligned} T(z)g &= \psi_1(\Lambda-z)(w'-u_0) + D(\psi_1,w'-u_0)-(\mathscr{A}_0-z)\tilde{u} &&\text{on}\quad \Omega\cap\mathbb{R}^3_+ \\ T(z)g &= (\Lambda-z)w' &&\text{on}\quad \Omega\cap\mathbb{R}^3_- \end{aligned}$$

where $\mathbb{R}^3_- = \{x \in \mathbb{R}^3; x_3 < 0\}$ and for $i = 1,2,3$

$$(5.52)\qquad D_i(\psi,v) = -\ \sigma_{ij}(v)\,\frac{\partial\psi}{\partial x_j} - \frac{\partial}{\partial x_j}[\lambda_0\delta_{ij}(\sum_{k=1}^{3} v_k\,\frac{\partial\psi}{\partial x_k})+\mu_0(v_i\,\frac{\partial\psi}{\partial x_j}+v_j\,\frac{\partial\psi}{\partial x_i})]$$

we have then the following lemma

Lemma 5.3

The function $T(z)$ *defined on* $\overline{\mathbb{C}^+}$ *with values in* $B(L^{2;s_1,s_2}(\Omega),H^1(\Omega))$ *is Hölder continuous of order* $\delta < \inf(1,s_1-1/2,s_2-1/2)$. $T(z)$ *is a compact operator in* $B(L^{2;s_1,s_2}(\Omega))$ *for every* z *in* $\overline{\mathbb{C}^+}$.

Proof of Lemma 5.3

Recall that g is in $L^{2;s_1,s_2}(\Omega)$ and let z and z' be in $\overline{\mathbb{C}^+}$. We then have

$$||w'(z)-w'(z')||_{H^1(\Omega^L)} \leq M||R_0^+(z)-R_0^+(z')||_{B(L^{2;s_1,s_2}(\mathbb{R}^3_+),H^{2;-s_1,-s_2}(\mathbb{R}^3_+))} \cdot |||g|||_{0;s_1,s_2} \tag{5.53}$$

where $R_0^+(z) = R_0(z)$ if $\operatorname{Im} z > 0$.

It then follows from theorem 3.7 that we have

$$||w'(z)-w'(z')||_{H^1(\Omega^L)} \leq M|z-z'|^\delta |||g|||_{0;s_1,s_2} \tag{5.54}$$

if $\delta < \inf(1,s_1-1/2,s_2-1/2)$.

From (5.54) we deduce that the first term of the r.h.s. of (5.51) has been estimated.

Let us remark that w' belongs to H^2 on $\{L < |x| < L+3\}$. It then follows that we have

$$||w'(z)-w'(z')||_{H^2(\{L<|x|<L+3,\mathbb{C}^3)} \leq M|z-z'|^\delta |||g|||_{0;s_1,s_2} \tag{5.55}$$

if $\delta < \inf(1,s_1-1/2,s_2-1/2)$.

Thus the second term of the r.h.s. of (5.51) is estimated. It is easy to estimate the third one because $\tilde{u}$ belongs to $H^3(\mathbb{R}^3)$. We get

(5.56) $$||T(z)g||_{H^1(\Omega)} \le M\, |||g|||_{0;s_1,s_2}$$

and

(5.57) $$||T(z)g-T(z')g||_{H^1(\Omega)} \le M|z-z'|^{\delta}|||g|||_{0;s_1,s_2} .$$

Let $I^+(a,b) = \{z \in \overline{\mathbb{C}^+} ; a \le \mathrm{Re}z \le b\}$ and suppose that $[a,b] \subset (0,\infty)$ we finally get for $\delta < \inf(1,s_1-1/2,s_2-1/2)$:

(5.58) $$||T(z)-T(z')||_{B(L^{2;s_1,s_2}(\Omega),H^1(\Omega))} \le M_\delta|z-z'|^{\delta}$$

for every (z,z') in $J^+(a,b) \times J^+(a,b)$.

The compactness of $T(z)$ considered as an operator in $B(L^{2;s_1,s_2}(\Omega))$ is the consequence of the Rellich theorem because the support of $T(z)g$ is contained in Ω^L and Ω satisfies the cone condition. Q.E.D.

Fourth step

Let

(5.59) $$u = (1-\psi_1)u_0 + \psi_1 w' - \tilde{u}$$

One easily checks that u is in $H^{1;-s_1,-s_2}(\Omega,A,\mathbb{C}^3)$ and that

(5.60) $$(\mathcal{A}-z)u = g + T(z)g .$$

Let

$$\overline{\mathbb{C}^+_c} = \{z \in \overline{\mathbb{C}^+} ; z \text{ is not a strictly positive eigenvalue of } A\} .$$

We then have :

Lemma 5.4

For any z *in* $\overline{\mathbb{C}_c^+}$, $I+T(z)$ *is invertible in* $B(L^{2;s_1,s_2}(\Omega))$.

The proof of lemma 5.4 is by contradiction.

Suppose there exists g in $L^{2;s_1,s_2}(\Omega)$, $g \neq 0$ and z in $\overline{\mathbb{C}_c^+}$ such that

$$(5.61) \qquad g + T(z)g = 0 .$$

From (5.60) we deduce that

$$(5.62) \qquad (\mathcal{A}-z)u = 0$$

where u is in $H^{1;-s_1,-s_2}(\Omega,A,\mathbb{C}^3)$.

If we prove that u is different from zero and is an eigenfunction of A associated with the eigenvalue z, this will be a contradiction because we have supposed that z belongs to $\overline{\mathbb{C}_c^+}$.

Now suppose $g \neq 0$ and $u = 0$. We then have $w' = 0$ in $\{x \in \Omega^L; |x| < L\}$. From (5.48) it follows that $g = 0$ in Ω_-^L. In particular w' can be extended to $B_+ = \{x \in \mathbb{R}_+^3; |x| < L+6\}$ by letting

$$(5.63) \qquad \begin{cases} \tilde{w}' = w' & \text{in } \Omega^L \\ \tilde{w}' = 0 & \text{in } B_+\backslash\Omega^L . \end{cases}$$

One easily verifies that $\tilde{w}'$ is in $H^2(B_+)$ and satisfies the free boundary condition on ∂B_+.

We have in Ω_+^L

$$(5.64)\qquad (A_0-\Lambda)w' = (A_L-\Lambda)w' = (\mathscr{A}_0-\Lambda)(\psi_2 u_0-\tilde{u}_0) \ .$$

Therefore

$$(5.65)\qquad \begin{aligned}(\mathscr{A}_0-\Lambda)(\tilde{w}'-\psi_2 u_0+\tilde{u}_0) &= 0 \quad \text{in} \quad \Omega_+^L \\ (\mathscr{A}_0-\Lambda)(\tilde{w}'-\psi_2 u_0+\tilde{u}_0) &= -(\mathscr{A}_0-\Lambda)u_0 \quad \text{in} \quad B_+\backslash\Omega_+^L \ .\end{aligned}$$

Let

$$(5.66)\qquad \omega = \tilde{w}' - \psi_2 u_0 + \tilde{u}_0 \ .$$

Let $\tilde{A}_0$ be the positive selfadjoint operator in $L^2(B_+,\mathbb{C}^3,\rho_0 dx)$ associated with $\mathscr{A}_0$ and the free boundary condition on ∂B_+. ω is then in the domain of $\tilde{A}_0$. It follows from (5.65) that $\tilde{A}_0\omega = \Lambda\omega$ in Ω_+^L and from (5.66) that $\omega = -u_0$ in $B_+\backslash\Omega_+^L$. Recall that $\mathscr{A}_0 u_0 = z u_0$ in $B_+\backslash\Omega_+^L$. We then have

$$\int_{B_+}(\lambda_0|\nabla.\omega|^2 + 2\mu_0\sum_{i,j=1}^{3}|\varepsilon_{ij}(\omega)|^2)dx = \int_{B_+}\tilde{A}_0\omega.\bar{\omega}\rho_0 dx$$

$$(5.67)\qquad = \int_{\Omega_+^L}\tilde{A}_0\omega.\bar{\omega}\rho_0 dx + \int_{B_+\backslash\Omega_+^L}\tilde{A}_0\omega.\bar{\omega}\rho_0 dx$$

$$= \Lambda\int_{\Omega_+^L}|\omega|^2\rho_0 dx - z\int_{B_+\backslash\Omega_+^L}|u_0|^2\rho_0 dx \ .$$

Thus the r.h.s. of (5.67) is a real number and $\omega = 0$ in Ω_+^L because we know that $\mathrm{Im}\Lambda$ is strictly negative and $\mathrm{Im}z$ is non negative.

Therefore

$$(5.68) \qquad \tilde{w}' = \psi_2 u_0 - \tilde{u}_0 \quad \text{in} \quad \Omega_+^L$$

$$(5.69) \qquad u = (1-\psi_1)u_0 + \psi_1\psi_2 u_0 - \psi_1\tilde{u}_0 - \tilde{u} \quad \text{in} \quad \Omega_+^L .$$

Remark that

$$(5.70) \qquad \psi_1\psi_2 = \psi_1 \quad \text{and} \quad \psi_1\tilde{u}_0 = 0 .$$

Therefore

$$(5.71) \qquad u = u_0 - \tilde{u} \quad \text{in} \quad \Omega_+^L$$

(5.71) implies that $\tilde{u}$ satisfies the free boundary condition on $\partial \mathbb{R}_+^3$. Thus h defined by (5.49) is equal to zero and $\tilde{u} = 0$. Finally $u = u_0 = 0$ in Ω_+^L. We then deduce that $\tilde{u}_0 = 0$ and $\tilde{w}' = 0$ in Ω_+^L. From (5.51) and (5.61) we conclude that $g = 0$. Then, if $\operatorname{Im} z > 0$, (5.59) and (5.62) show that u belongs to $D(A)$ and is an eigenfunction of A associated with the eigenvalue z. This is impossible because A is a selfadjoint operator.

Now suppose $z = \sigma^2 > 0$ and $\sigma^2 \in \overline{\mathbb{C}_c^+}$. In this case u is an eigenfunction of A associated with the eigenvalue σ^2. Indeed $I = \langle\tilde{g}, R_0^+(\sigma^2)\tilde{g}\rangle$ is a real number. This is proved exactly in the same way as in the last part of the first proof of theorem 5.2. We then have

$$(5.72) \qquad \tau_P(\sigma)\tilde{g} = \tau_{SV}(\sigma)\tilde{g} = \tau_{SV}^0(\sigma)\tilde{g} = \tau_{SH}(\sigma)\tilde{g} = \tau_R(\sigma)\tilde{g} = 0 .$$

It follows from (5.72) and theorem 4.1 that u_0 is in $L^2(\mathbb{R}_+^3, \mathbb{C}^3, \rho_0 dx)$. Therefore u is <u>in</u> $D(A)$. But this is impossible because we have supposed that σ^2 is in $\overline{C_c^+}$. In conclusion we have $g = 0$ and the lemma 5.4 is proved by using the compactness of $T(z)$. Q.E.D.

It is now easy to conclude the second proof of theorem 5.2.

Given f in $L^{2;s_1,s_2}(\Omega)$ and z in $\overline{C_c^+}$ we first solve the following equation

$$(5.73) \qquad f = (1 + T(z))g \; .$$

It results from lemma 5.4 that there exists an unique g in $L^{2;s_1,s_2}(\Omega)$ solution of (5.73).

From g and the first two steps and from (5.59) we construct u. If $\mathrm{Im}z > 0$, u_0 is in $H^2(\mathbb{R}^3_+,\mathbb{C}^3)$ and u is in $D(A)$. From (5.60) we then have

$$(5.74) \qquad u = R(z)f \; .$$

If $\mathrm{Im}z = 0$ we define $R^+(z)$ by

$$(5.75) \qquad u = R^+(z)f \; .$$

Finally by using (5.73), (5.74), (5.75) and lemma 5.3 we easily prove for $\delta < \inf(1,s_1-1/2,s_2-1/2)$ that

$$(5.76) \qquad \|R^+(z)-R^+(z')\|_{B(L^{2;s_1,s_2}(\Omega),L^{2;-s_1,-s_2}(\Omega))} \le M_\delta |z-z'|^\delta$$

for some positive constant M depending of δ and for $(z,z') \in J^+(a,b) \times J^+(a,b)$ where $[a,b]$ is contained in $\overline{C_c^+}$. In (5.76) we have let $R^+(z) = R(z)$ if $\mathrm{Im}z > 0$.

Of course by considering $\overline{C_c^-} = \{z \in \mathbb{C};\ \mathrm{Im}z \le 0 \ , \ z \neq 0$ and z is not an eigenvalue of $A\}$ we get a similar result for $R^-(z)$.

This concludes the second proof of theorem 5.2. Q.E.D.

Finally by using theorem 4.1 and theorem 5.2 and as in [6] we prove the following theorem.

Theorem 5.5

(i) A *has no continuous singular spectrum.*

(ii) If [a,b] *is a compact interval contained in* $(0,\infty)$ A *can only have a finite number of eigenvalues in* [a,b] *and each of these eigenvalues has a finite multiplicity.*

(iii) Let u *be any eigenfunction of* A *associated with a strictly positive eigenvalue. We then have*

$$u \in \bigcap_{s<2} L^2(\Omega,\mathbb{C}^3,(1+x_3)^s dx) \ . \tag{5.77}$$

In conclusion let us note that for perturbations of A_0 in $\mathbb{R}^2_+$ the results given in theorems 5.2 and 5.5 are the same as for perturbations of A_0 in $\mathbb{R}^3_+$.

REFERENCES

[1] J.D. ACHENBACH, Wave propagation in elastic solids. North Holland, Amsterdam, 1973.

[2] J.D. ACHENBACH, A.K. GANTESEN and H. McMAKEN. Ray methods for waves in elastic solids with applications to scattering by cracks. Pitman, 1982.

[3] S. AGMON. Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola Norm. Sup. Pisa Ser. IV, 2 (1975), 151-218.

[4] K. AKI and P.G. RICHARDS. Quantitative seismology. Theory and methods. Vol. I and Vol. II. Freeman, 1980.

[5] B.A. AULD. Acoustic fields and waves in solids. Vol. I and Vol. II. Wiley Interscience, 1973.

[6] Y. DERMENJIAN and J.C. GUILLOT. Théorie spectrale de la propagation des ondes acoustiques dans un milieu stratifié perturbé. 1984, to be published in the "Journal of Differential Equations".

[7] Y. DERMENJIAN and J.C. GUILLOT. Les ondes élastiques dans un demi-espace isotrope. Développement en fonctions propres généralisées. Principe d'absorption limite. C.R.A.S., 300 (1985), 93-96.

[8] Y. DERMENJIAN and J.C. GUILLOT. Théorie de la diffusion des ondes élastiques dans un demi-espace isotrope perturbé. C.R.A.S., 301 (1985), 617-619.

[9] N. DUNFORD and J.T. SCHWARTZ. Linear operators. Part II, Spectral theory. Interscience Publishers, New York, 1963.

[10] G. DUVAUT and J.L. LIONS. Les inéquations en mécanique et en physique. Dunod, Paris, 1972.

[11] A.G. ERINGEN and E.S. SUHUBI. Elastodynamics. Vol. II Linear theory. Academic Press, 1975.

[12] D.M. EIDUS. The principle of limiting absorption, A.M.S., Transl. (2) 47 (1965), 157-191.

[13] D.M. EIDUS. The principle of limiting amplitude, Russian Math. Surveys 24 (1969), 97-167.

[14] J. GOBERT. Une inéquation fondamentale de la théorie de l'élasticité, Bull. Soc. Roy. Sci. Liège, 31è année, n° 3-4 (1962), 182-191.

[15] J.C. GUILLOT and C.H. WILCOX. Spectral analysis of the Epstein operator, Proc. Roy. Soc. Edinburgh 80A(1978), 85-98.

[16] J.C. GUILLOT. Théorie spectrale et développement en fonctions propres pour les équations de Maxwell dans une couche diélectrique Rapport n° 385, I.N.R.I.A., 78153 Le Chesnay, France, 1985.

[17] J.C. GUILLOT. Existence and uniqueness of a Rayleigh surface wave propagating along the free boundary of a transversely isotropic elastic half space. To be published in Math. Math. Appl. Sci.

[18] L. HORMANDER. The analysis of linear partial differential operators II. Springer, Berlin, 1983.

[19] T. KATO. Perturbation theory for linear operators. Springer, New York, 1966.

[20] A. MAJDA. Outgoing solutions for perturbations of $-\Delta$ with applications to spectral and scattering theory. J. Diff. Eq. 16 (1974) 515-547.

[21] S. MIZOHATA. The theory of partial differential equations. Cambridge University Press, Londres, 1973.

[22] R. PHILLIPS. On the exterior problem for the reduced wave equation. A.M.S. Summer Institute in Partial Differential Equations, Berkeley, 1971.

[23] M. REED and B. SIMON. Methods of Modern Mathematical Physics. Vol. II Fourier Analysis, Self Adjointness. Academic Press, New York, 1975.

[24] M. REED and B. SIMON. Methods of Modern Mathematical Physics. Vol. III. Scattering Theory. Academic Press, New York, 1978.

[25] M. REED and B. SIMON. Methods of Modern Mathematical Physics. Vol. IV. Analysis of Operators. Academic Press, New York, 1978.

[26] M. ROSEAU. Asymptotic wave theory. North Holland, 1976.

[27] M. SCHECHTER. Operator Methods in Quantum Mechanics. Elsevier North Holland, 1981.

[28] J.R. SCHULENBERGER. Elastic waves in the half space $\mathbb{R}^2_+$. J. Diff. Eq. 29 (1978), 405-438.

[29] R. WEDER. Spectral and scattering theory in perturbed stratified fluids, II, Transmission problems and exterior domains. Communicaciones Técnicas IIMAS-UNAM. Serie Naranja n° 368, 1984.

[30] C.H. WILCOX. Scattering theory for the d'Alembert equation in exterior domains. Lecture Notes in Mathematics, n° 442, Springer Berlin, 1975.

[31] C.H. WILCOX. Spectral analysis of the Pekeris operator. Arch. Rational Mech. Anal. 60 (1976), 259-300.

[32] C.H. WILCOX. Sound propagation in stratified media. Applied Mathematical Sciences, Vol. 50, Springer, 1984.

[33] C.H. WILCOX. Transient electromagnetic wave propagation in a dielectric waveguide. Istituto Nazionale di Alta Matematica, Symposia Matematica XVIII, 1976, 239-277.

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